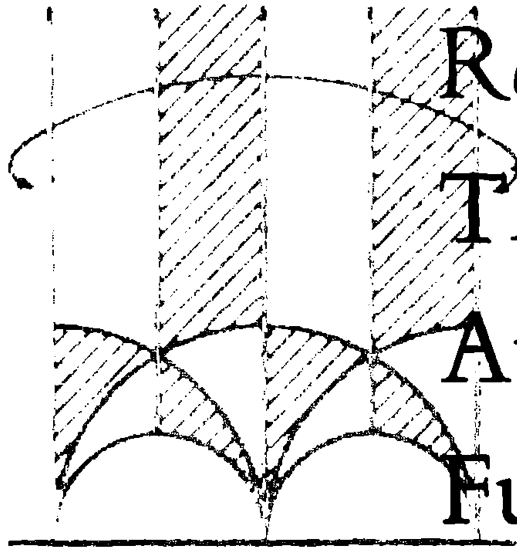
I.M. GELFAND

REPRESENTATION THEORY AND AUTOMORPHIC FUNCTIONS

Note

The theory of group representations has given us a new understanding of classical results in the theory of automorphic functions and has made it possible to attack the problems of this theory on a wider scale and obtain a number of new and profound results. The language of the theory of adeles—a recently developed branch of mathematics—plays an important role. The book contains many new ideas and results that have so far been accessible only in mathematical journals. Therefore, the book should appeal to various circles of readers interested in contemporary mathematics. It may be recommended to students in advanced courses, to Ph.D. candidates and to research workers in pure mathematics.



Representation Theory and Automorphic Functions

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TRANSLATED FROM THE RUSSIAN BY K. A. HIRSCH

W. B. SAUNDERS COMPANY

Philadelphia · London · Toronto

Preface

The classical theory of automorphic functions, created by Klein and Poincaré, was concerned with the study of analytic functions in the unit circle that are invariant under a discrete group of transformations. Since the unit circle can be regarded as a Lobachevskii plane in the Poincaré model, we may say that the classical theory of automorphic functions dealt with the study of functions analytic on the Lobachevskii plane and invariant under a discrete group of motions of the plane.

In the subsequent development of the theory of automorphic functions the papers of Hecke, Siegel, Selberg, and a number of other investigators played an essential part. In particular, papers by Godement, Maass, Roelcke, Peterson, and Langlands cover one or another aspect of the connection between automorphic functions and the theory of groups. Another very interesting direction in the theory of automorphic functions can be found in works of Ahlfors and Bers.

The whole development of the theory of automorphic functions pointed forcefully to the necessity of a group-theoretical approach. Recently many of the ideas of the theory have been linked with arbitrary Lie groups and their discrete subgroups.

The connection between the theory of group representations and the theory of automorphic functions was made particularly precise in the last ten or twenty years, in the context of the development of the theory of infinite-dimensional representations of groups. Although this connection was perceived much earlier (for example, in papers of Klein and Hecke), a true understanding was achieved only after the construction of the theory of infinity-dimensional representations of Lie groups.

One of the first papers to establish this relationship was by Gel'fand and Fomin, in which the concepts of representation theory were linked with the theory of dynamical systems and the theory of automorphic functions. The connection of automorphic functions with dynamical systems already occurs, in essence, in earlier papers of Hopf on dynamical systems.

Apart from the theory of infinite-dimensional representations of Lie groups, which had received a strong impetus in the last twenty years (in papers of Gel'fand and Naimark, Harish-Chandra, Gel'fand and Graev, and others), an important part in the construction of the modern theory of automorphic functions was the creation of the theory of algebraic groups by Chevalley, Borel, Harish-Chandra, Tits, and others.

Perhaps one of the most remarkable ideas that have arisen in recent years is that of the group of adeles. In the process of writing this book the authors have convinced themselves how natural many concepts become when they are applied to the group of adeles and its discrete subgroup of principal adeles.

The book consists of three chapters. In the first chapter we consider problems of representation theory and the theory of automorphic functions connected with a Lie group and a discrete subgroup of it. Although the individual questions of this chapter are of a general character, the main results refer to the group of real matrices of order 2 and its discrete subgroups. In particular, in this chapter we give an account, in the language of representation theory, of the remarkable results of Selberg (Selberg's trace formula).

In the second chapter we construct the theory of representations of the group of matrices of order 2 with elements from an arbitrary locally compact topological field. The well-studied theory of representations of the group of complex matrices and the group of real matrices arises here as a special case. Many facts of representation theory become more conceptual in this general approach. We also mention that the special functions over an arbitrary field, which arise naturally in this theory, are closely related to interesting functions in the theory of numbers (Gauss sums, Kloostermann sums, and others).

The third chapter is devoted to a study of the groups of adeles and the natural homogeneous spaces that arise in connection with these groups. Since it is assumed that the reader is not acquainted with the theory of adeles, the first two sections provide an expository account of the basic ideas of this theory.

With the group of adeles there is connected a remarkable homogeneous space (the space of cosets relative to the subgroup of principal adeles), which has been the main object of study in all papers concerned with adeles. But whereas these papers were

devoted to the study of the homogeneous space itself, the computation of its volume (the Tamagawa number), and so forth, we study here the space of functions on this homogeneous space (see **§ 4, 6, 7**). From this point of view the fundamental paper of Tate, in which he gives a derivation of the functional equation of the Riemann Zeta-function by means of adeles, can be regarded as an analogue (for the case of matrices of order 1) of the study of representations that we pursue here. Many of our results were also obtained later by other methods by Godement, whose work was very useful in writing **§ 4** of this chapter.

The last three sections are devoted to the beginnings of the general theory for adele groups of an arbitrary algebraic reductive group. A fundamental role in this theory is played by a certain group of automorphisms of the function space that forms a representation of the Weyl group. Symmetry with respect to this group is a veritable key to relations of the type of the functional equation for the Riemann Zeta-function. These automorphisms are closely connected with the so-called horospherical maps. The fact that much of the material in these sections is of quite recent origin inevitably leaves its mark on the character of the exposition itself, which is frequently complicated.

The authors hope, however, that the additional burden the reader assumes in coping with these sections is perhaps compensated by the fact that, if he so wishes, he may participate in the work on these far from completely answered questions.

The book can be read independently of the preceding volumes of the series *Generalized Functions*. However, conceptually it is closely connected with the theory of generalized functions and especially with the contents of volume 5, which deals with analogous problems in other material. It can be regarded as a natural extension of the fifth volume.

The authors are deeply indebted to A. A. Kirillov, who has accepted the arduous task of editing the book and of writing one of the sections (Appendix to Chapter II) in which he expounds his own new results.

Since sending the manuscript to the printers the authors have become acquainted with a preprint of an interesting new paper by Langlands, the material of the Summer School on the Theory of Algebraic Groups, and a paper by Moore. In these papers the reader will find additional information on the material of this book.

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Contents

HOMOGENEOUS SPACES WITH A DISCRETE STABILITY GROUP

1

§ 1. GENERALITIES

1. Homogeneous Spaces and Their Stability Subgroups. We begin with some general definitions.

Let X be a topological space and G a topological group. We say that G is a *group of transformations* or a *group of motions* of X if to every element g of G there corresponds a one-to-one and bicontinuous transformation

$$x \to xg$$

of X into itself. Here we assume that the following conditions hold:

1. To the unit element e of G corresponds the identity transformation, that is, $xe = x$ for every x in X.

2. $(xg_1)g_2 = x(g_1g_2)$ for every x in X and any g_1, g_2 in G.

3. The function $f(x, g) = xg$ that assigns to every pair $x \in X$ and $g \in G$ the point $xg \in X$ is a continuous function of the pair x and g.

A space X with a group of motions G is called *homogeneous* if every point x of it can be carried by motions into every other point. We also say then that G *acts transitively* on X.

We recall how all homogeneous spaces on which a given group G acts transitively can be described in terms of G itself.

Let X be a homogeneous space with the group of motions G,

In X we fix a point x_0. With every point x in X we associate the set of transformations that carry x_0 into x. Let us see what this set is. To begin with we examine the transformations that carry x_0 into x_0. Obviously they form a closed subgroup Γ of G. This subgroup is called the *stability group* of x_0. Next, if g is one of the transformations that carry x_0 into x, then the set of all transformations carrying x_0 into x is the right coset Γg of Γ.

So we have established a correspondence between the points of the homogeneous space X and the right cosets of Γ. This correspondence is one-to-one.

Observe that the set of cosets $\Gamma \setminus G$ is naturally endowed with the structure of a topological space: the neighborhoods of the coset Γg are the images of the neighborhoods of g under the map $G \to \Gamma \setminus G$.

The action of g in the original space X clearly corresponds to the multiplication of the right cosets by g in the space $\Gamma \setminus G$.

Thus, *every homogeneous space with the group of motions G can be obtained by the following construction. We take a subgroup Γ of G. With the points of X we associate the right cosets Γg of G with respect to Γ. The action corresponding to the element g_0 of G is defined as multiplication of the cosets on the right by g_0.*

This space of right cosets will always be denoted as follows:

$$X = \Gamma \setminus G.$$

We have established that every homogeneous space X with the group of motions G can be identified with the space of cosets $\Gamma \setminus G$, where Γ is the stability group of a point x_0 in X. Here the choice of the point x_0 itself is completely arbitrary. It is easy to see that the stability groups of distinct points are conjugate; for if an element g carries x_0 into x, then the stability group of x is the group $g^{-1}Gg$. Consequently, the spaces $\Gamma \setminus G$ and $g^{-1}\Gamma g \setminus G$ connected with conjugate subgroups should be identified with each other.

Our main object in this chapter is to study the homogeneous spaces $\Gamma \setminus G$, where G is the group of real (unimodular) matrices of order 2

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with the determinant $\alpha\delta - \beta\gamma = 1$, and Γ is a discrete subgroup of G.

2. The Connection Between Homogeneous Spaces $X = \Gamma \setminus G$ and Riemann Surfaces. There is a close connection between the spaces $X = \Gamma \setminus G$, where G is the group of real unimodular matrices of order 2 and Γ is a discrete subgroup of G, and Riemann surfaces. For $\Gamma \setminus G$ can be interpreted as a fiber space whose base is some Riemann surface and whose fiber is a circle.

We begin with the case in which Γ is the trivial subgroup, that is, with the group space G itself.

We consider all possible conformal self-transformations of the upper half-plane $\mathrm{Im}\, z > 0$ of the plane of the complex variable z. It is well known that every such transformation can be described by a real matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with determinant 1. The transformation has the following form:

$$z' = \frac{\alpha z + \gamma}{\beta z + \delta}.$$

Here the product of two matrices corresponds to the product of the related transformations.

Obviously the two matrices g_1 and g_2 define the same conformal self-transformation of the half-plane $\mathrm{Im}\, z > 0$ only if $g_2 = \pm g_1$. Thus, *the group G_0, which is obtained from G by identifying the matrices g and $-g$, is isomorphic to the group of all conformal transformations of the half-plane* $\mathrm{Im}\, z > 0$.

We define a linear element on a Riemann surface as a pair: a point and a direction given at this point.

We shall show that the elements of G_0 may be interpreted as linear elements on the half-plane $\mathrm{Im}\, z > 0$. Let us fix a linear element l_0 on the half-plane, and let l be any other linear element. It is known that there exists one and only one conformal self-transformation g of the half-plane that carries l_0 into l. So we have established the required one-to-one correspondence between the elements g of G_0 and the linear elements l on the half-plane $\mathrm{Im}\, z > 0$.

Hence, the space of elements of G in which g and $-g$ are identified may be interpreted as the space of all linear elements in the upper half-plane $\mathrm{Im}\, z > 0$. In other words, it is a fiber space whose base is the half-plane $\mathrm{Im}\, z > 0$ and whose fiber is a circle. (The circle is identified with the set of directions at a point.)

Now we shall give a similar interpretation for the space $X = \Gamma \backslash G$, where Γ is a discrete subgroup of G. We assume that Γ contains the element $-e$ and that the elements $\gamma \neq \pm e$ of Γ do not leave any point of the half-plane $\mathrm{Im}\, z > 0$ fixed.†

We shall show presently that the space $X = \Gamma \backslash G$ may be interpreted as the space of linear elements on a Riemann surface. In other words, it is a fiber space whose base is a certain Riemann surface and whose fiber is a circle.

On the half-plane $\mathrm{Im}\, z > 0$ we identify the points that are carried into one another by transformations of Γ. As a result we

† The last condition is equivalent to the condition that α does not contain elements of finite order other than $\pm e$.

obtain a certain Riemann surface $\mathscr{D}$, that is, a one-dimensional (not necessarily compact) complex manifold. Now we consider the linear elements on the half-plane $\operatorname{Im} z > 0$ and identify those that are carried into one another by transformations of Γ. It is easy to see that the space so obtained may be interpreted as the space of all linear elements on the Riemann surface $\mathscr{D}$.

Here the assumption that a transformation $\gamma \neq \pm e$ of Γ has no fixed points is essential. For if some $\gamma \neq \pm e$ leaves a point z_0 fixed, we must identify those linear elements at z_0 that are carried into each other by γ.

Let us show that this space of linear elements on the Riemann surface is isomorphic to the original space $X = \Gamma \setminus G$.

We know that linear elements on the half-plane $\operatorname{Im} z > 0$ may be treated as elements of the group G_0 obtained from G by identifying matrices g and $-g$. In this treatment each linear element on $\mathscr{D}$ forms a set γg, where γ ranges over Γ, that is, a coset of Γ in G. So the space of linear elements on $\mathscr{D}$ turns out to be identical with the space of cosets $\Gamma \setminus G$.

We have obtained the following interpretation of the space $X = \Gamma \setminus G$.

Let G be the group of real unimodular matrices of order 2, Γ a discrete subgroup of G containing the element $-e$ and not containing elements of finite order other than $\pm e$. On the half-plane $\operatorname{Im} z > 0$ we identify points that are obtained from each other by linear-fractional transformations from Γ. We obtain a certain Riemann surface $\mathscr{D}$. The space of cosets $X = \Gamma \setminus G$ is the space of linear elements on this Riemann surface $\mathscr{D}$ and so is a fiber space whose base is $\mathscr{D}$ and whose fiber is a circle.

A similar result is true when Γ contains elements of finite order. However, the homogeneous structure of the fiber space then breaks down at individual points.

The upper half-plane $\operatorname{Im} z > 0$ is a universal covering for the Riemann surface $\mathscr{D}$. But we are not concerned here with Riemann surfaces having as universal covering either the full sphere or the punctured sphere. The spaces of linear elements for such Riemann surfaces can be constructed more simply than in our case.

The space of linear elements on a Riemann surface has a number of advantages compared with the Riemann surface itself. The main advantage is that the space of linear elements is homogeneous. Its group of automorphisms is the group G_0 of linear-fractional transformations. Besides, as a rule the Riemann surface has a rather poor supply of admissible automorphisms.

For example, it is not difficult to show that the number of automorphisms on every compact Riemann surface for which the upper half-plane $\operatorname{Im} z > 0$ is a universal covering is finite. Let $\mathscr{D}$ be one of these Riemann surfaces, and let Γ be the discrete group, corresponding to $\mathscr{D}$, of automorphisms of the half-plane

Im $z > 0$. We examine an automorphism γ of $\mathscr{D}$. This automorphism can be extended in a natural way to a conformal transformation g' of the whole upper half-plane Im $z > 0$, that is, to a fractional-linear transformation. Obviously, this conformal transformation g' commutes with Γ, that is,

$$g'\Gamma g'^{-1} = \Gamma$$

Thus, g' belongs to the normalizer N of Γ. Our aim is to prove that the coset space $\Gamma \setminus G$ contains only a finite number of elements. Let us show that N is a discrete subgroup of G and, consequently, that $\Gamma \setminus N$ is a discrete space.

We consider an arbitrary hyperbolic element γ of Γ; without loss of generality we may assume that γ is a diagonal matrix, $\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Furthermore, let γ' be any other element of Γ, but not a diagonal matrix. We assume that N is not a discrete subgroup. Then we can find in N a sequence of elements g_n converging to the unit matrix. We examine the elements $\gamma_n = g_n^{-1}\gamma g_n$ and $\gamma_n' = g_n^{-1}\gamma' g_n$. They belong to Γ and at the same time converge to γ and γ', respectively. Since Γ is discrete, from a certain n onward we must have $g_n^{-1}\gamma g_n = \gamma$, $g_n^{-1}\gamma' g_n = \gamma'$, that is, g_n commutes with γ and γ'. From the fact that g_n commutes with the diagonal matrix γ it follows that g_n is itself a diagonal matrix. But this is impossible, because then g_n cannot commute with γ'.

So we have shown that $\Gamma \setminus N$ is a discrete space. On the other hand, from the compactness of $\Gamma \setminus G$ it follows that $\Gamma \setminus N$ is also compact. Consequently, $\Gamma \setminus N$ contains only a finite number of elements, as required.

3. The Fundamental Domain of a Discrete Group Γ. Let Y be a topological space in which a discrete group Γ of homeomorphisms γ acts:

$$y \to \gamma y$$

To say that Γ is discrete means that for every $y \in Y$ the set of points γy, where γ ranges over Γ, has no accumulation points in Y. We shall always assume that Γ acts on Y *effectively*. This means that for every $\gamma \neq e$ there is a point y in Y for which $\gamma y \neq y$. (The points y for which $\gamma y = y$ for at least one $\gamma \neq e$ will henceforth be called fixed points.)

Let us give examples of such spaces Y.

1. Y is a topological group and Γ a discrete subgroup of it, Γ acting on Y as a group of left translations.

2. Y is the upper half-plane (Im $z > 0$) of the plane of the complex variable z, and Γ is a discrete group of conformal self-transformations of the upper half-plane.

We now introduce the concept of a *fundamental domain* in Y relative to a group Γ. A fundamental domain in Y relative to a group Γ is defined as an open set $F \subset Y$ satisfying the following two conditions:

1. For arbitrary $\gamma_1 \neq \gamma_2$ the sets $\gamma_1 F$ and $\gamma_2 \bar{F}$, where $\bar{F}$ is the closure of F, have no common elements.

2. The union of the sets $\gamma \bar{F}$, where γ ranges over Γ, is the whole space Y.

These conditions can be rephrased in the following way: Every point y of Y can be represented in the form

$$y = \gamma x, \tag{1}$$

where $\gamma \in \Gamma$, $x \in \bar{F}$. This representation is unique for almost all points y; namely, if $y = \gamma_1 x_1 = \gamma_2 x_2$, where $\gamma_1, \gamma_2 \in \Gamma$ and $x_1 \in F$, $x_2 \in \bar{F}$, then $\gamma_1 = \gamma_2$, $x_1 = x_2$. Thus, the points y for which the decomposition (1) may not turn out to be unique form the set $\Gamma(\bar{F} \setminus F)$.

Observe that a fundamental domain relative to a group Γ is by no means uniquely defined by these conditions. In particular, if F is a fundamental domain, then every translation γF of it, where $\gamma \in \Gamma$, is also a fundamental domain.

We shall now describe a method of constructing a fundamental domain under some simple additional conditions on the space Y and the group Γ.

We assume that Y is a locally compact metric space, where the metric $\rho(y_1, y_2)$ in Y satisfies the following condition:

For any two points y_0 and y_1 we can find a third point y_2 such that

$$\rho(y_0, y_2) = \rho(y_2, y_1) = \tfrac{1}{2}\rho(y_0, y_1). \tag{2}$$

We call such a metric ρ *intrinsic.*

An example of a space with an intrinsic metric is an ordinary sphere on which the distance between points is measured by the angle on a great circle. Note that if the distance between points on the sphere is defined differently, namely, as the length of a chord joining them, then the metric is not intrinsic.

It is easy to check that in a complete space with an intrinsic metric any two points y_0, y_1 can be joined by an arc $y(t)$, $0 \leq t \leq 1$, $y(0) = y_0$, $y(1) = y_1$, such that

$$\rho(y(t_1), y(t_2)) = (t_2 - t_1)\rho(y_0, y_1), \qquad 0 \leq t_1 \leq t_2 \leq 1.$$

In particular, such a space is connected.

It can also be shown that in a locally compact space with an intrinsic metric every bounded and closed set is compact. We assume that Γ preserves the metric of the space Y, that is,

$$\rho(\gamma y_0, \gamma y_1) = \rho(y_0, y_1) \tag{3}$$

for arbitrary $y_0, y_1 \in Y$ and $\gamma \in \Gamma$.

Now we can construct a fundamental domain relative to Γ.

In Y we select a point y_0 that is not a fixed point. We consider the set F of those points y for which

$$\rho(y_0, y) < \rho(\gamma y_0, y) \tag{4}$$

for every $\gamma \neq e$ in Γ.

Let us show that F is a fundamental domain relative to Γ.

Since Γ is discrete, it follows that F is an open set. Let $y_1 \in F$ and $\rho(y_0, y_1) = d$. In Y we examine the neighborhood of y_0

$$U_{3d} = \{y: \rho(y_0, y) < 3d\}.$$

This neighborhood is compact and hence there is only a finite number of elements $\gamma_1, \ldots, \gamma_n$, $\gamma_i \neq e$, in Γ such that $\gamma_i y_0$ belongs to U_{3d}. Obviously for the remaining elements $\gamma \in \Gamma$ we have $\rho(\gamma y_0, y_1) \geq 2d$. We consider the ε-neighborhood of y_1

$$U_\varepsilon = \{y: \rho(y_1, y) < \varepsilon\}.$$

It is easy to check that for $\varepsilon < \min\left(\frac{d}{2}, \frac{d_i - d}{2}, i = 1, \ldots, n\right)$, $d_i = \rho(\gamma_i y_0, y_1)$, this neighborhood belongs to F. Consequently F is open.

Next we show that F satisfies condition 1 that is, the sets F and $\gamma \bar{F}$, $\gamma \neq e$, do not intersect. For let $y \in \bar{F}$. Then for every γ in Γ we have

$$\rho(y_0, y) \leq \rho(\gamma^{-1} y_0, y).$$

Since $\rho(y_0, y) = \rho(\gamma y_0, \gamma y)$ and $\rho(\gamma^{-1} y_0, y) = \rho(y_0, \gamma y)$ we see that

$$\rho(\gamma y_0, \gamma y) \leq \rho(y_0, \gamma y).$$

This inequality means that for $\gamma \neq e$ the element γy does not belong to F; thus, the set $\gamma \bar{F}$, $\gamma \neq e$, does not intersect F.

Finally, we show that F satisfies condition 2, that is, every point $y \in Y$ can be represented in the form

$$y = \gamma x,$$

where $\gamma \in \Gamma$, $x \in \bar{F}$.

It is easy to check that the closure $\bar{F}$ of F consists of all those points y for which

$$\rho(y_0, y) \leq \rho(\gamma y_0, y), \qquad \gamma \in \Gamma.$$

Let y be an arbitrary point of Y. Then we can find a $\gamma_0 \in \Gamma$ such that

$$\rho(\gamma_0 y_0, y) \leq \rho(\gamma y_0, y) \tag{5}$$

for every $\gamma \in \Gamma$. (Otherwise some neighborhood of y contains infinitely many points γy_0; but this is impossible, since Γ is discrete.)

Since the metric ρ is invariant, it follows from (5) that

$$\rho(y_0, \gamma_0^{-1} y) \leq \rho(\gamma y_0, \gamma_0^{-1} y)$$

for every $\gamma \in \Gamma$. Thus, the point $\gamma_0^{-1} y = x$ belongs to $\bar{F}$, hence $y = \gamma_0 x$, where $\gamma_0 \in \Gamma$, $x \in \bar{F}$.

So we have shown that our set F is in fact a fundamental domain relative to Γ.

4. Discrete Groups with a Compact Fundamental Domain. There is much interest in those discrete groups of transformations Γ for which the closure $\bar{F}$ of a fundamental domain is a compact set. First we give a condition for compactness of $\bar{F}$. We assume that there exists a compact subset $K \subset Y$ such that

$$Y = \Gamma K,$$

that is, every $y \in Y$ can be represented in the form

$$y = \gamma k,$$

where $\gamma \in \Gamma$, $k \in K$.

Then a fundamental domain F relative to Γ such as we have constructed above has a compact closure.

We note that the closure $\bar{F}$ of F consists of all points y of Y for which

$$\rho(y_0, y) \leq \rho(\gamma y_0, y), \qquad \gamma \in \Gamma, \qquad y_0 \in F. \tag{6}$$

Let us assume that $\bar{F}$ is not compact. Then $\bar{F}$ is not bounded and so we can find in it a sequence of points y_n such that $\rho(y_0, y_n) \to \infty$. To show that this is impossible, we represent y_n in the form

$$y_n = \gamma_n k_n,$$

where $\gamma_n \in \Gamma$, $k_n \in K$. By (6) we have

$$\rho(y_0, y_n) \leq \rho(\gamma_n y_0, \gamma_n k_n),$$

and since the metric ρ is invariant,

$$\rho(y_0, y_n) \leq \rho(y_0, k_n).$$

Since K is a compact set, the sequence $\rho(y_0, k_n)$ is bounded. But then the sequence $\rho(y_0, y_n)$ is also bounded, and this contradicts our hypothesis.

We now discuss properties of discrete groups of transformations Γ and of their fundamental domains F, when $\bar{F}$ is a compact set.

The following two propositions hold:

1. *There exists a finite number of elements* $\gamma_1, \ldots, \gamma_n$ *of* Γ *such that* $\bar{F}$ *can be given by the finite number of inequalities*

$$\rho(y_0, y) \leq \rho(\gamma_i y_0, y), \qquad i = 1, \ldots, n.$$

2. *The group* Γ *is finitely generated.*

Proof of 1. By the compactness of $\bar{F}$ there exists a number $c > 0$ such that

$$\rho(y_0, y) \leq c \tag{7}$$

for every y in $\bar{F}$. We consider the elements γ in F such that

$$\rho(y_0, y) = \rho(\gamma y_0, y) \tag{8}$$

for at least one element $y \in \bar{F}$. There is only a finite number of such

elements.† For it follows from (7) and (8) that

$$\rho(y_0, \gamma y_0) \leq 2c; \tag{9}$$

since Γ is discrete, the inequality (9) can be satisfied only by a finite number of elements γ. We denote the elements $\gamma \neq e$ satisfying (8) by $\gamma_1, \ldots, \gamma_n$.

Next we show that $\bar{F}$ can be given by the finite number of inequalities

$$\rho(y_0, y) \leq \rho(\gamma_i y_0; y) \qquad i = 1, \ldots, n. \tag{10}$$

Let us assume the contrary: then there exists a point y' satisfying (10), but not belonging to $\bar{F}$.

We denote by E the compact set of points y satisfying (10) and

$$\rho(y_0, y) \leq c_1,$$

where $c_1 = \rho(y_0, y') + c$. This set contains $\bar{F}$ and the point y', which does not belong to $\bar{F}$. We show that E is a connected set. Let $y_1 \in E$. We consider a continuous arc $y(t)$, $0 \leq t \leq 1$, joining y_0 and y_1 and such that

$$\rho(y(t_1), y(t_2)) = (t_2 - t_1)\,\rho(y_0, y_1), \qquad 0 \leq t_1 \leq t_2 \leq 1.$$

It is easy to see that all points of this arc satisfy (10) and consequently belong to E. Thus E is connected.‡

Next we show that $\bar{F}$ is open in E. Indeed, $\bar{F}$ can be given in E by an infinite number of inequalities

$$\rho(y_0, y) < \rho(\gamma y_0, y), \tag{11}$$

where $\gamma \neq e, \gamma_1, \ldots, \gamma_n$. But all these inequalities, except possibly a finite number of them, hold in the whole set E (for they hold for all γ for which $\rho(y_0, \gamma y_0) > 3c_1$). Consequently $\bar{F}$ can in fact be given in E by a finite number of inequalities of the form (11), and hence $\bar{F}$ is open in E. So $\bar{F}$ is a closed and open subset of E. But since E is connected, we must have $\bar{F} = E$; this contradicts the assumption we have made and proves proposition 1.

Proof of 2. We consider an open bounded set U containing $\bar{F}$. We shall show that $\bar{U}$ can be covered by a finite number of sets $\gamma\bar{F}$. If we consider the contrary, then we can find in $\bar{U}$ a sequence of elements y_n of the form $y_n = \gamma_n x_n$, where $\gamma_n \in \Gamma$, $x_n \in \bar{F}$ and the γ_n

† The set of these elements is not empty. Otherwise $\bar{F} = F$, that is, F is an open and closed set. This contradicts the fact that the space is connected.

‡ For we have $\rho(\gamma_i y_0, y(t)) \geq \rho(\gamma_i y_0, y_1) - \rho(y_1, y(t))$; consequently, since

$$\rho(\gamma_i y_0, y_1) \geq \rho(y_0, y_1),$$
$$\rho(y_1, y(t)) = (1 - t)\,\rho(y_0, y_1),$$

we obtain

$$\rho(\gamma_i y_0, y(t)) \geq t\rho(y_0, y(1)) = \rho(y_0, y(t)).$$

are distinct. Since $\bar{U}$ and $\bar{F}$ are compact sets, we may assume without loss of generality that $y_n \to y$, $x_n \to x$. But then it is obvious that $\gamma_n x \to y$. This is impossible, since Γ is discrete.

So we have shown that $\bar{U}$ can be covered by a finite number of sets $\gamma\bar{F}$, say by $\gamma_1\bar{F}, \ldots, \gamma_n\bar{F}$. We examine the subgroup Γ' generated by the γ_i and show that $\Gamma' = \Gamma$, so that Γ is finitely generated. We consider the set

$$Y' = \bigcup_{\gamma'\in\Gamma'} \gamma'\bar{F}.$$

By construction of Γ', this set contains together with every $\gamma'\bar{F}$ also a neighborhood $\gamma' U$ of it. Consequently Y' is open. But then the set

$$\gamma Y' = \bigcup_{\gamma'\in\Gamma'} \gamma\gamma'\bar{F},$$

is also open, where γ is an arbitrary element of Γ. The sets $\gamma Y'$ cover the whole space Y. It is easy to see that two such sets $\gamma_1 Y'$ and $\gamma_2 Y'$ either coincide or are disjoint. If at least two of them are distinct, then Y is a union of pairwise disjoint open sets γY; this is impossible, because Y is connected. So all the $\gamma Y'$ coincide, hence $Y = Y' = \bigcup_{\gamma'\in\Gamma'} \gamma'\bar{F}$. Consequently, for any $\gamma \in \Gamma$ whatsoever, the set $\gamma\bar{F}$ is contained in Y'; but then $\gamma F = \gamma' F$ for some $\gamma' \in \Gamma'$ and so $\gamma = \gamma'$.

So we have shown that Γ coincides with the finitely generated subgroup Γ'.

Another proposition on discrete groups of transformations with a compact fundamental domain can be stated.

3. *If Y is a simply-connected space, then Γ can be given by a finite number of defining relations among its generators.*

A proof of this proposition can be found, for example, in the paper of Weil [70].

So far we have discussed properties of discrete groups of transformations of an arbitrary locally compact space Y. Now we consider the case when Γ is a discrete subgroup of a locally compact group G. We show that if the space $\Gamma \setminus G$ is compact, then the set of elements in G that are conjugate to any fixed $\gamma \in \Gamma$ is closed in Γ.

Let $g_i^{-1}\gamma g_i$, $\gamma \in \Gamma$, be a sequence convergent to some $g \in G$. We show that g is also conjugate to γ.

We represent the g_i in the form $g_i = \gamma_i u_i$ where $\gamma_i \in \Gamma$, $u_i \in \bar{F}$, and F is a fundamental domain relative to Γ. By hypothesis, $\bar{F}$ is a compact set. Therefore, without loss of generality, we may assume that the sequence u_i converges to an element $u \in \bar{F}$. Then the equation

$$\lim_{i\to\infty} (u_i^{-1}\gamma_i^{-1}\gamma\gamma_i u_i) = g$$

implies that

$$\lim_{i\to\infty} \gamma_i^{-1}\gamma\gamma_i = ugu^{-1}.$$

But since Γ is a discrete subgroup, the convergent sequence $\gamma_i^{-1}\gamma\gamma_i$ must be stationary for sufficiently large indices i. Hence, for sufficiently large i we have $\gamma_i^{-1}\gamma\gamma_i = ugu^{-1}$, that is, the elements g and γ are conjugate. This completes the proof.

We apply this result to the case when G is the group of real unimodular matrices of order 2. The elements $g \neq \pm e$ of G fall into three classes: elliptic elements (matrices with complex eigenvalues), hyperbolic elements (matrices with distinct real eigenvalues) and parabolic elements (matrices with multiple eigenvalues equal to $+1$ or -1).

Clearly, the set of elements of G that are conjugate to a parabolic element g is not closed. The closure of this set contains one of the matrices e or $-e$.

We conclude: *If for a discrete subgroup Γ of the group G of real unimodular matrices of order 2 $\Gamma \setminus G$ is a compact space, then Γ consists only of elliptic and hyperbolic elements.*

5. The Structure of a Fundamental Domain in the Lobachevskii Plane. Let Y be a Lobachevskii plane and Γ a discrete subgroup of motions on Y. In this subsection we study properties of a fundamental domain corresponding to Γ in the case when the volume $v(\Gamma \setminus Y)$ of this fundamental domain is finite.

Here we can interpret the Lobachevskii plane Y as the upper half-plane

$$\operatorname{Im} z > 0$$

of the plane of the complex variable z; the motions in Y are then all the linear-fractional self-transformations

$$z' = \frac{\alpha z + \gamma}{\beta z + \delta}, \qquad \alpha\delta - \beta\gamma = 1$$

with real coefficients α, β, γ, δ.

The real line

$$\operatorname{Im} z = 0$$

can be treated as the family of points at infinity of the Lobachevskii plane.

We recall how a fundamental domain F corresponding to a discrete subgroup of motions Γ can be constructed. On Y we select a point z_0 that is not a fixed point (that is, $\gamma z_0 \neq z_0$ for $\gamma \neq 1$). Then a fundamental domain F with center at z_0 is given by a system of inequalities

Observe that F is bounded by geodesic arcs. For each of the equations

$$\rho(z_0, z) = \rho(\gamma z_0, z),$$

which define the boundary, is the equation of a geometric locus of points that are equidistant from the two given points z_0 and γz_0. But it is well known that such a geometric locus of points is a geodesic.

Thus, *the boundary of a fundamental domain F is a polygonal curve A (possibly disconnected and possibly consisting of an infinite number of sides) formed by geodesic arcs.* The polygon bounded by this curve is a star domain, because F contains, together with every point z, the entire geodesic arc joining the points z_0 and z.

We shall now prove the following theorem, due to Siegel [65], on the number of sides of the polygon.

If the area of the fundamental domain F is finite, then the number of geodesic arcs that form the boundary of F is finite.

Note that for a domain F with compact closure the theorem has already been proved under **3**; therefore, we need to consider only a domain without compact closure.

The main step of the proof is an estimate for the angles ω at the vertices of F. We show that

$$\sum_{\omega} (\pi - \omega) \leq I + 2\pi, \tag{2}$$

where the sum is taken over all the vertices of F that do not lie on the line at infinity, and I is the volume of F. Let us now prove the inequality (2).

We join all the vertices of A to z_0 by geodesics and consider the triangles so obtained. Let

$$\ldots A_m, A_{m+1}, \ldots, A_n, \ldots$$

be a connected set of arcs of A with the vertices $\ldots a_m, a_{m+1}, \ldots, a_{n+1}, \ldots$ (Figure 1). To be definite we assume that this set is unbounded in both directions. By α_k, β_k, γ_k we denote the angles of

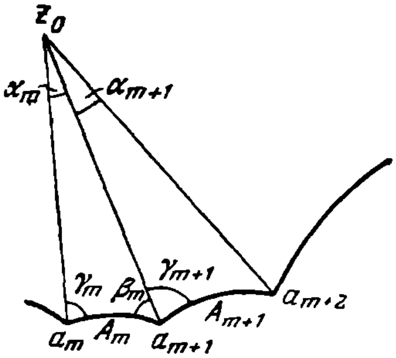

Figure 1

the triangle with the side A_k, and by ω_k the angle between A_k and A_{k+1}; so we have

$$\omega_k = \beta_k + \gamma_{k+1}.$$

Now we can estimate the angles ω_k. We use the formula for the area of a triangle with angles α, β, γ in a Lobachevskii plane:

$$I = \pi - \alpha - \beta - \gamma.$$

By this formula the area $I(A_k)$ of the triangle with the side A_k is

$$I(A_k) = \pi - \alpha_k - \beta_k - \gamma_k.$$

Consequently,

$$\sum_{k=m}^{n} \alpha_k + \sum_{k=m}^{n} I(A_k) = \pi - \gamma_m - \beta_n + \sum_{k=m}^{n-1} (\pi - \omega_k). \tag{3}$$

But the left-hand side of this equation is bounded because $\sum I(A_k) \leq v(F)$, where $v(F)$ is the area of F, and $\sum \alpha_k \leq 2\pi$; hence, the right-hand side is also bounded. From this it follows that the series $\sum (\pi - \omega_k)$ converges, and that the limits $\lim\limits_{m \to -\infty} \gamma_m = \gamma_{-\infty}$ and $\lim\limits_{n \to +\infty} \beta_n = \beta_\infty$ exist.

Now we show that $\pi - \gamma_{-\infty} - \beta_\infty \geq 0$. For $a_k \to \infty$ as $k \to \infty$ (because only a finite number of arcs of A can be at an unbounded distance from z_0); hence, $\rho(z_0, a_k) > \rho(z_0, a_{k-1})$ for infinitely many values of k. But for these values of k we then have $\gamma_k > \beta_k$. On the other hand, since $\beta_k + \gamma_k \leq \pi$, we have $\beta_k \leq \dfrac{\pi}{2}$. Consequently $\beta_\infty \leq \dfrac{\pi}{2}$. Similarly, we see that $\gamma_{-\infty} \leq \dfrac{\pi}{2}$. So we have shown that $\pi - \gamma_{-\infty} - \beta_\infty \geq 0$.

When we pass to the limit as $m \to -\infty$, $n \to +\infty$ in (3) and bear in mind the inequality $\pi - \gamma_{-\infty} - \beta_\infty \geq \infty$, we see that

$$\sum_{k=-\infty}^{+\infty} \alpha_k + \sum_{k=-\infty}^{+\infty} I(A_k) \geq \sum_{k=-\infty}^{+\infty} (\pi - \omega_k). \tag{4}$$

The inequality (4) was obtained under the assumption that the connected set of the A_k is unbounded in both directions. By similar arguments we can verify that the same inequality is valid also in the other cases when the connected set of the A_k is bounded in at least one direction.

Adding all these inequalities we obtain the required estimate:

$$2\pi + I \geq \sum_{\omega} (\pi - \omega), \tag{5}$$

where the sum is taken over all the vertices of F (at a finite distance from z_0) and I is the volume of F.

On the basis of this estimate we shall now show that the number of vertices of F at a finite distance from z_0 is finite. Let a be one of the vertices, and $a^{(1)} = a, a^{(2)}, \ldots$ all the vertices of F equivalent to a:

$$a^{(i)} = \gamma_i a, \qquad \gamma_i \in \Gamma.$$

By $\omega^{(i)}$ we denote the angles at the vertices $a^{(i)}$. It is easy to verify that if a is not a fixed point for $\gamma \in \Gamma$, $\gamma \neq 1$, then

$$\omega^{(1)} + \omega^{(2)} + \cdots = 2\pi. \tag{6}$$

But if a is a fixed point of order n (that is, the number of elements $\gamma \in \Gamma$ that leave a invariant is n), then

$$\omega^{(1)} + \omega^{(2)} + \cdots = \frac{2\pi}{n}. \tag{6'}$$

To see this let us find all the displacements of F that abut on a. Obviously, these are the domains $\gamma\gamma_i^{-1}F$, where γ ranges over the n elements of Γ that leave a in its place. Since the domain $\gamma\gamma_i^{-1}F$ has the angle $\omega^{(i)}$ at the vertex a, and since the sum of all the angles at a is equal to 2π, we have

$$\omega^{(1)} + \omega^{(2)} + \cdots = \frac{2\pi}{n}.$$

Obviously, the equation (6′) is incompatible with (5) if F has an infinite number of vertices.

It remains to show that the number of vertices of F on the line at infinity is also finite.

We choose any N vertices of F on the line at infinity: $B_1, \ldots, B_N$. Clearly, we can construct a polygon bounded by a finite number of geodesic arcs and lying inside F such that its vertices at infinity are the points $B_1, \ldots, B_N$.

By a passage to the limit it is easy to verify that for the area I_1 of this polygon the following formula holds:

$$\sum_{\omega} (\pi - \omega) = 2\pi + I_1,$$

where the sum is taken over all the vertices of the polygon, and ω are the angles at the vertices. Since $\omega = 0$ for a vertex at infinity, we deduce that

$$\pi N \leq 2\pi + I_1 \leq 2\pi + v(F).$$

So the number N is bounded.

This shows that if the volume of F is finite, then the number of its vertices is finite. In particular, so is the number of vertices of F on the real axis E:

$$\operatorname{Im} z = 0$$

Now we shall study properties of the vertices of F on the real axis E.

First of all we observe that *if F is not compact, then it has at least one vertex on E.* Consider all possible geodesics starting at z_0; each geodesic uniquely determines a direction l at z_0. We denote by $\tau(l)$ the length of the arc of the geodesic inside F. The number $\tau(l)$ may be equal to ∞, in which case the geodesic lies entirely within F. Obviously $\tau(l)$ is a continuous function of l for those l for which $\tau(l) < \infty$. Therefore, if $\tau(l) < \infty$ for all l, then the function $\tau(l)$ is bounded. But then F is a compact domain. Consequently, if F is not compact, there exists a direction l for which $\tau(l) = \infty$. We take one of these directions l. The intersection with the real axis E of the geodesics that start from z_0 in the direction l is a vertex of F. So we have shown that F actually has vertices on the real axis E.

We now show that *for every vertex b of F on E there exists an element $\gamma \in \Gamma$, $\gamma \neq \pm 1$, that leaves b fixed.*

Let b be one of the vertices of F on E. We examine all the displacements γF of F having their vertex at b. Clearly there is an infinite number of such displacements γF, which we now describe. Let $b^{(1)} = b, b^{(2)}, \ldots, b^{(n)}$ be the vertices of F equivalent to b:

$$b^{(k)} = \gamma_k b, \qquad \gamma_k \in \Gamma, \qquad k = 1, \ldots, n.$$

By what we have shown above, the number of these vertices is finite. Clearly every displacement of F having b as its vertex is of the form

$$\gamma\gamma_i^{-1}F,$$

where γ ranges over the elements of Γ that leave b fixed. Since there is an infinite set of such displacements, whereas γ_i ranges over only a finite set, there must exist an infinite set of elements γ that leave the point b fixed.

So we have shown that there must exist an element $\gamma \in \Gamma$, $\gamma \neq \pm 1$, that leaves b fixed. We now show that *every such element γ is parabolic.*†

Let us assume the contrary: γ is not a parabolic element. We consider the geodesic $z(t)$, $0 \leq t \leq \infty$, $z(0) = z_0$, joining the points z_0 and b. This geodesic lies entirely within F, therefore

$$\rho(z_0, z(t)) < \rho(\gamma z_0, z(t)), \qquad 0 \leq t < \infty. \tag{7}$$

Through z_0 we lay the horocycle ω orthogonal to $z(t)$. This horocycle can be represented in the form of a circle touching the real axis at b (Figure 2). Since by assumption γ is not parabolic, the

† A transformation $\gamma \in \Gamma$ is called parabolic if it is given by a matrix with multiple eigenvalues equal to $+1$ or to -1.

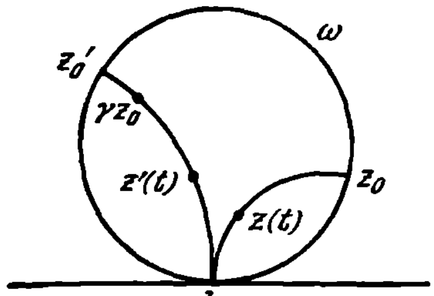

Figure 2

point γz_0 does not belong to this horocycle (otherwise the transformation γ carries the whole horocycle into itself, and only parabolic transformations have this property). Without loss of generality we may assume that γz_0 lies inside the horocycle; otherwise we replace γ by γ^{-1}. We lay a geodesic $z'(t)$ through the points γz_0 and b and measure the arc length from the point of intersection z_0' of this geodesic with the horocycle ω. Then we have

$$\rho(z_0, z(t)) = \rho(z_0', z'(t)) = \rho(z_0', \gamma z_0) + \rho(\gamma z_0, z'(t)). \qquad (8)$$

But, as is well known, $\rho(z(t), z'(t)) \to 0$ as $t \to \infty$; therefore, the distance $\rho(\gamma z_0, z'(t))$ differs by an arbitrarily small amount from $\rho(\gamma z_0, z(t))$ for sufficiently large t. Consequently, by (8) we find that for sufficiently large t

$$\rho(z_0, z(t)) > \rho(\gamma z_0, z(t)).$$

This inequality contradicts (7). So the assumption that γ is not parabolic is false.

Let us state the final result. If $v(\Gamma \setminus Y) < \infty$, then there exists a fundamental domain F bounded by a finite number of geodesic arcs. Here the vertices of F on the real axis are parabolic points; this means that for each of them there exists a parabolic transformation $\gamma \in \Gamma$, $\gamma \neq \pm 1$, that keeps this vertex fixed.

We shall now show that there exists a fundamental domain F satisfying the following additional condition: the vertices of F on the real axis are pairwise inequivalent.

It is enough to show the following proposition. Let b be a vertex of a fundamental domain F on the real axis such that there exists vertices $b^{(1)} = \gamma_1 b, \ldots, b^{(p)} = \gamma_p b$ equivalent to b. Then we can construct another fundamental domain that has fewer vertices on the real axis than F has.

For let l be one of the sides of the polygon F starting from b. It is easy to verify that there exists an element $\gamma \in \Gamma$ with the following properties:

1. γ carries a certain vertex $b^{(k)}$ into b.

2. γ carries one of the sides l', l'' of the polygon F starting from b_k, into l.

Let $F_k \subset F$ be a triangle formed by the geodesics l', l'', and a third geodesic. We consider the domain $F' = (F \setminus F_k) \cup \gamma F_k$, obtained from F by deleting F_k and adding the triangle γF_k. Clearly this F' is a fundamental domain relative to Γ. By construction it has on the real axis the same vertices as F except for $b^{(k)}$. This completes the proof of the proposition.

From our description of the fundamental domain F it follows immediately that it can be split into subdomains of simpler structure. For let $F(b)$, where b is a parabolic point, denote the triangle bounded by two geodesics starting from b and the horocycle ω touching the real axis at b. Then we have

$$F = \sum_{k=1}^{n} F(b_k) + F_0,$$

where the sum is taken over all the parabolic vertices of F, and F_0 is a compact set.

It is easy to verify that each of the domains $F(b_k)$ has the following properties. Let $F(b_k)$ be bounded by two geodesics l and l' and the horocycle ω. Then the geodesics l and l' are equivalent to each other, that is, they are carried into one another by some transformation $\gamma \in \Gamma$ leaving b_k fixed; also, every point inside ω can be carried into $F(b_k)$ by some transformation $\gamma \in \Gamma$ leaving b_k fixed.

In **§ 6** essential use will be made of such a decomposition of a fundamental domain.

§ 2. REPRESENTATIONS OF A GROUP G INDUCED BY A DISCRETE SUBGROUP

With every discrete subgroup Γ of a locally compact group G we associate a certain collection of unitary representations of G, the so-called induced representations. These representations are reducible. Our task is to decompose them into irreducible representations or, what is the same, to find their spectrum.

We attack this problem in the case where $X = \Gamma \setminus G$ is a compact space. In **§ 2.3** we shall show that in this case the representations of G connected with a subgroup Γ have discrete spectra of finite multiplicity. In other words, they split into a discrete sum of irreducible representations each of which occurs in the decomposition with a finite multiplicity. In **§ 2.4** we shall obtain the trace formula, which enables us to classify all the

irreducible representations occurring in the decomposition. Applications of the trace formula to some concrete groups will be given in **§ 5.**

1. Definition of Induced Representations. Let G be a locally compact topological group. With each discrete subgroup Γ of G we associate a collection of unitary representations of G.

We begin with a description of the simplest of these representations. It is constructed in the space of functions $f(x)$ on $X = \Gamma \setminus G$, having integrable square modulus:

$$\|f\|^2 = \int |f(x)|^2 \, dx < \infty,$$

where dx is the invariant measure on X. The representation associates with every element g of G an operator $T(g)$ of the following form:†

$$T(g)f(x) = f(xg). \tag{1}$$

(We recall that xg denotes the point of X into which x is carried by g.)

The operators $T(g)$ are unitary; this follows immediately from the fact that the measure dx is invariant under the motions $x \to xg$.

We call (1) the representation *generated by the homogeneous space* $\Gamma \setminus G$.

The general construction of a representation of G induced by a subgroup Γ consists in the following:

Let $\chi(\gamma)$ be a finite-dimensional unitary representation of Γ, acting in a space V.‡ We consider the Hilbert space $H(\chi)$ of all measurable vector functions $f(g)$ on G with values in V satisfying the following two conditions:

1. $$f(\gamma g) = \chi(\gamma) f(g) \tag{2}$$

for every $\gamma \in \Gamma$.

2. $$(f, f) \equiv \int_X [f, f] \, dx < \infty, \tag{3}$$

where $[f_1, f_2]$ is the inner product in the finite-dimensional space V. The representation associates with every element g_0 of G an operator $T(g_0)$ of the following form:

$$T(g_0)f(g) = f(gg_0). \tag{4}$$

It is easy to verify that these operators are unitary.

† It would be more correct to write $(T(g)f)(x)$. However, to simplify the notation we always omit the extra pair of parentheses.

‡ A similar construction is, of course, also possible for infinite-dimensional unitary representations of Γ.

We say that the representation (4) is *induced by the subgroup* Γ (or, more accurately, induced by the given representation $\chi(\gamma)$ of Γ).

Observe that the representation (1) is a special case of this construction in which $\chi(\gamma)$ is the unit representation. Then $f(g)$ is a scalar function so that the representation space V of $\chi(\gamma)$ is one-dimensional. The condition (2) then takes the form $f(\gamma g) = f(g)$.

This means that the functions $f(g)$ are constant on the cosets $\Gamma \setminus G$ so that they can be regarded as functions given on the homogeneous space $X = \Gamma \setminus G$. We now indicate another realization of the representation $T(g)$ induced by Γ. Let F be a fundamental domain in G relative to Γ. This means that every element g of G can be represented in the form

$$g = \gamma x, \tag{5}$$

where $\gamma \in \Gamma$, $x \in \bar{F}$, and the decomposition (5) is unique for all elements g with the exception of a set of smaller dimension.

Obviously every vector function $f(g)$ from the representation space $H(\chi)$ is uniquely determined by its values on $\bar{F}$; conversely, every function on F can be uniquely extended to a function satisfying the condition (2) on the whole group G.

In this way we arrive at a new realization of the space $H(\chi)$. In this model the elements of $H(\chi)$ are all possible vector functions $f(x)$, $x \in F$, assuming values in the representation space V of $\chi(\gamma)$ and satisfying the condition

$$(f,f) \equiv \int_F [f,f]\, dg < \infty \tag{6}$$

$[f,f]$ is the scalar product in V.

Note that (6) does not depend on the choice of the fundamental domain; this follows from the fact that the expression $[f(g),f(g)]$ is preserved when g is replaced by γg, where $\gamma \in \Gamma$. In this model the representation operator $T(g)$ is given by the following formula:

$$T(g)f(x) = \chi(\gamma)f(x'), \tag{7}$$

where $\gamma \in \Gamma$ and $x' \in F$ are defined by the relation

$$xg = \gamma x'. \tag{8}$$

With the discrete subgroup Γ we can associate yet another important representation of G. Consider the set of all subgroups Γ_i of finite index in G. Let $T_i(g)$ be the representation of G generated by the homogeneous space $X_i = \Gamma_i \setminus G$. As we know, this representation acts in the space $L_2(X_i)$ of functions of integrable square on X_i.

Observe now that if $\Gamma_i \subset \Gamma_j$, then we have a natural imbedding $L_2(X_j) \subset L_2(X_i)$. For functions from $L_2(X_j)$ can be regarded as

functions on G that are constant on the cosets of Γ_j; but then they are also constant on the cosets of Γ_i, that is, they belong to $L_2(X_i)$.

Hence it follows that we can construct another Hilbert space H, which is the direct limit of the spaces $L_2(X_i)$. This space can be described as follows. We index the groups Γ_i by means of the natural numbers and we set $\Gamma'_0 = \Gamma$, $\Gamma'_i = \Gamma_1 \cap \cdots \cap \Gamma_i$ $(i = 1, 2, \ldots)$, $X'_i = \Gamma'_i \setminus G$. Then we have $\Gamma'_i \subset \Gamma'_{i-1}$, and so $L_2(X'_{i-1}) \subset L_2(X'_i)$. We denote by H_i the orthogonal complement of the subspace $L_2(X'_{i-1})$ in $L_2(X'_i)$. Then H is the direct sum of the Hilbert spaces $L_2(X)$ and of $H_1, H_2, \ldots$.

There is a natural definition for a representation of G in H, because one is defined in each of the spaces $L_2(X), H_1, H_2, \ldots$.

Some of the results concerning spectra relative to $L_2(X)$ remain valid for the space H.

Let us assume, for example, that $X = \Gamma \setminus G$ is a compact space. Since the subgroups Γ_i have finite index in G, the spaces $X_i = \Gamma_i \setminus G$ are then also compact. As we shall show later, in this case each of the spaces $L_2(X_i)$ splits into the direct sum of a countable number of invariant irreducible subspaces, and each of these subspaces occurs in $L_2(X_i)$ with a finite multiplicity. In other words, the spectrum of the representation in $L_2(X_i)$ is discrete and of finite multiplicity.

Clearly in this case the representation in H also splits into a countable direct sum of irreducible representations.

It would be very interesting to study this decomposition in detail. For example, is it true to say, at least in the particular case when G is the group of real matrices of order 2, that the irreducible representations occurring in H form in a certain sense an everywhere dense set in the space of all representations? Is the multiplicity with which an irreducible representation occurs in this decomposition finite?

2. The Operators T_φ. An important role in the theory of representations is played by operators of the form

$$T_\varphi = \int \varphi(g) T(g)\, dg, \tag{1}$$

where $T(g)$ is the representation operator, $\varphi(g)$ is a certain function on the group, and integration is with respect to the invariant measure dg on G.†

The integral (1) necessarily converges when $\varphi(g)$ is a continuous finite function on G (or a function that decreases sufficiently fast at infinity).‡

† Throughout we assume that the measure dg is *two-sided invariant*, that is, $dg = d(gg_0) = d(g_0 g)$ for every element $g_0 \in G$.

‡ that is, of compact support (Tr.)

It is easy to verify that if φ_1 and φ_2 are finite continuous functions, and λ_1 and λ_2 complex numbers, then

$$T_{\gamma_1\varphi_1+\gamma_2\varphi_2} = \lambda_1 T_{\varphi_1} + \lambda_2 T_{\varphi_2}, \tag{2}$$

$$T_{\varphi_1 * \varphi_2} = T_{\varphi_1} T_{\varphi_2} \tag{3}$$

where $\varphi_1 * \varphi_2$ is the convolution. Thus, *the correspondence $\varphi \to T_\varphi$ is a representation of the algebra of finite continuous functions $\varphi(g)$ with multiplication defined as convolution.*

We mention also that if T is a unitary representation, then

$$T_{\varphi *} = T^*_\varphi, \tag{4}$$

where $\varphi^*(g) = \overline{\varphi(g^{-1})}$. From (3) and (4) it follows that $T_{\varphi * \varphi *}$ is a self-adjoint positive definite operator.

The transition from the representation operator $T(g)$ to T_φ is convenient because the latter sometimes turns out to be a completely continuous integral operator (or an integral operator with a Hilbert–Schmidt kernel). This allows us to apply classical results in the theory of integral operators to the investigation of representations T.

In this subsection we discuss the unitary representation of a locally compact group G induced by a discrete subgroup Γ. We show that *if $X = \Gamma \setminus G$ is a compact space, then for every continuous finite function $\varphi(g)$ on G*

$$T_\varphi = \int \varphi(g) T(g)\, dg \tag{5}$$

is a completely continuous integral operator.

Proof. We give an explicit expression for T_φ. The representation $T(g)$ of G connected with Γ is given by a certain unitary representation $\chi(\gamma)$ of Γ acting in a finite-dimensional space V. The representation $T(g)$ acts in the space $H(\chi)$ of vector functions on G with values in V and satisfying the condition

$$f(\gamma g) = \chi(\gamma) f(g), \qquad \gamma \in \Gamma, \qquad g \in G. \tag{6}$$

The representation operator $T(g)$ is given by the formula

$$T(g_0) f(g) = f(g g_0). \tag{7}$$

By means of (5) and (7) we obtain the following formula for T_φ:

$$T_\varphi f(g_1) = \int \varphi(g) f(g_1 g)\, dg.$$

Making the change of variable $g_1 g = g'$, we obtain

$$T_\varphi f(g_1) = \int_G \varphi(g_1^{-1} g') f(g')\, dg' = \int_F \Big(\sum_{\gamma \in \Gamma} \varphi(g_1^{-1} \gamma g_2) \chi(\gamma) f(g_2) \Big)\, dg_2,$$

where F is a fundamental domain relative to Γ. So we see that T_φ is an integral operator whose kernel is

$$K(g_1, g_2) = \sum_{\gamma \in \Gamma} \varphi(g_1^{-1}\gamma g_2)\chi(\gamma). \tag{8}$$

Note that the summation in (8) is taken, in fact, over a finite set γ. For we may assume that $g_1^{-1}\gamma g_2$ belongs to a fixed compact set for all γ, because $\varphi(g)$ is a finite function. On the other hand, g_1 and g_2 also belong to a compact set, namely the closure of the fundamental domain F. Therefore γ also belongs to some compact set. But the collection of all γ is discrete, and so γ in (8) can range only over a finite set.

From the remark just made it follows that the kernel $K(g_1, g_2)$ of T_φ is a continuous function of g_1, g_2. Hence T_φ is a completely continuous operator in $H(\chi)$.†

We now show that T_φ is completely continuous for a class of functions wider than that of finite functions. Let us consider a continuous function $\varphi(g)$ satisfying the following condition: there exists a nonnegative summable function $\varphi_1(g)$ on G and a compact neighborhood U of the unit element of G such that for every $g_0 \in G$

$$|\varphi(g_0)| \le \int_U \varphi_1(g_0 g)\, dg. \ddagger \tag{9}$$

Next we show that the series (8) for the kernel $K(g_1, g_2)$ of T_φ (which need no longer be finite) is absolutely convergent.

Indeed, for fixed g_1, g_2 and γ we have the inequality

$$|\varphi(g_1^{-1}\gamma g_2)| < \int_U \varphi_1(g_1^{-1}\gamma g_2 g)\, dg. \tag{10}$$

Observe that the sets $g_1^{-1}\gamma g_2 U$ corresponding to γ and γ' for fixed g_1 and g_2 overlap if and only if $\gamma'^{-1}\gamma \in g_2 U U^{-1} g_2^{-1}$. Since Γ is discrete, the compact set $g_2 U U^{-1} g_2^{-1}$ contains only a finite number N of elements of Γ. Hence it follows that for fixed g_1 and g_2 not more than N of these sets $g_1^{-1}\gamma' g_2 U$ can intersect $g_1^{-1}\gamma g_2 U$. But then we

† Here we use the following well-known fact: let H be the space of functions $f(x)$ given on a compact set X and such that $\int |f(x)|^2\, dx < \infty$. If $K(x, y)$ is a continuous function on $X \times X$, then an integral operator on H with kernel $K(x, y)$ is completely continuous.

‡ It is easy to see that finite functions satisfy this condition. For if $\varphi(g)$ is a finite function, then we can take for U any compact neighborhood of the unit element, and $\varphi_1(g)$ can be defined, for example, by the formula

$$\varphi_1(g_0) = \frac{1}{\operatorname{mes} U} \max_{g \in U} |\varphi(g_0 g^{-1})|$$

have by (10)

$$\sum_{\gamma\in\Gamma} |\varphi(g_1^{-1}\gamma g_2)| \le N \int_G \varphi_1(g)\, dg$$

that is, $\sum_{\gamma\in\Gamma} \varphi(g_1^{-1}\gamma g_2)$ is an absolutely convergent series. From this it follows immediately that the series (8) for $K(g_1, g_2)$ is absolutely convergent.

By similar arguments we can verify that the series for K converges uniformly in g_1 and g_2, when g_1 and g_2 belong to a fixed compact domain in G and, in particular, when $g_1, g_2 \in F$. Since all the terms of the series for $K(g_1, g_2)$ are continuous, $K(g_1, g_2)$ is a continuous function of g_1 and g_2.

Thus, *if $X = \Gamma \setminus G$ is a compact space, then for every continuous function $\varphi(g)$ satisfying* (9) *the operator $T_\varphi = \int \varphi(g) T(g)\, dg$ is a completely continuous integral operator.*

3. The Discreteness of the Spectrum of the Induced Representation in the Case of a Compact Space $X = \Gamma \setminus G$. As before, let G be a locally compact topological group, and Γ a discrete subgroup of it. We shall now prove the following proposition.

THEOREM. *If $X = \Gamma \setminus G$ is a compact space, then the representation $T(g)$ of G induced by Γ splits into a discrete sum of a countable number of irreducible unitary representations, each of finite multiplicity.*

In other words, the spectrum of the representation of G in $X = \Gamma \setminus G$ is discrete and of finite multiplicity.

To prove the theorem we consider the operator

$$T_\varphi = \int \varphi(g) T(g)\, dg,$$

where $\varphi(g)$ is a continuous finite function on G. In **§ 2.2** we have shown that these operators are completely continuous. Therefore the proof of the theorem reduces to the proof of the following proposition.

LEMMA. *If, for a unitary representation $g \to T(g)$ of a locally compact group G in a space H, the operator $T_\varphi = \int \varphi(g) T(g)\, dg$ is completely continuous for every finite continuous function $\varphi(g)$, then H splits into the sum of a countable number of invariant subspaces on which the representations of G are irreducible and of finite multiplicity.*

Proof of the Lemma. We consider continuous finite functions $\varphi(g)$ satisfying the additional condition

$$\varphi(g) = \overline{\varphi(g^{-1})}, \tag{1}$$

The operator $T_\varphi = \int \varphi(g) T(g)\, dg$ corresponding to such a function is a *self-adjoint*, completely continuous integral operator. For we

have

$$T_\varphi^* = \int \overline{\varphi(g)}\, T^*(g)\, dg = \int \overline{\varphi(g)}\, T(g^{-1})\, dg = \int \overline{\varphi(g^{-1})}\, T(g)\, dg,$$

so that, by (1), $T_\varphi^* = T_\varphi$.

Consequently, T_φ has a countable discrete spectrum, and all the eigenvalues $\lambda \neq 0$ of T_φ are of finite multiplicity. (See, for example, Smirnov [67].) Thus, the representation space H splits into the direct sum of subspaces

$$H = H_\varphi \dotplus \sum_{k=1}^{\infty} H_{\varphi,k}, \tag{2}$$

where H_φ is the subspace of all vectors f with the eigenvalue 0, that is, those for which $T_\varphi f = 0$; $H_{\varphi,k}$ is the subspace of all eigenvectors of T_φ with the given eigenvalue $\lambda_k \neq 0$. All the spaces $H_{\varphi,k}$ are finite-dimensional. Obviously, a similar decomposition holds for every invariant subspace $H' \subset H$, namely:

$$H' = H'_\varphi + \sum_{k=1}^{\infty} H'_{\varphi,k}, \tag{3}$$

where $H'_\varphi \subset H_\varphi$, $H'_{\varphi,k} \subset H_{\varphi,k}$.

Now we consider all possible subspaces $H_{\varphi,k}$, where φ ranges over the continuous finite functions satisfying (1). Let H_1 be the minimal subspace containing all the $H_{\varphi,k}$. We show that H_1 coincides with the whole space H.

Suppose the contrary. We choose a vector $f \neq 0$ from the orthogonal complement of H_1. This vector is orthogonal to the $H_{\varphi,k}$, and so by (2) it belongs to H_φ. In other words, $T_\varphi f = 0$ for every function φ. We show that this is impossible.

From the definition of the representation of a group it follows that the sequence of vectors $T(g)f$ converges to f when g converges to the unit element. Thus, for every $\varepsilon > 0$ we can find a neighborhood U of the unit element for which

$$\| T(g)f - f \| < \varepsilon \, \| f \|$$

when $g \in U$. We choose a continuous finite function $\varphi(g)$ satisfying apart from (1) also the following two conditions:

1. $\varphi(g)$ is concentrated in U and assumes only real nonnegative values.

2. $\int \varphi(g)\, dg = 1$.

For such a function $\varphi(g)$ we have

$$T_\varphi f - f = \int_U \varphi(g)(T(g)f - f)\, dg$$

Consequently,

$$\|T_\varphi f - f\| \leq \int_U \varphi(g) \, \|T(g)f - f\| \, dg < \varepsilon \, \|f\|.$$

Since $T_\varphi f = 0$ by our assumption, we then have $\|f\| < \varepsilon \, \|f\|$ for every $\varepsilon > 0$, which is impossible.

So we have shown that the minimal space containing all the finite-dimensional spaces $H_{\varphi,k}$ coincides with the whole space H. Hence, from (3) it follows that every invariant subspace of H has a nonzero intersection with at least one $H_{\varphi,k}$. On the basis of this fact we construct a decomposition of H into the direct sum of invariant irreducible subspaces.

We fix an arbitrary subspace $H_{\varphi,k}$ and examine the intersection of $H_{\varphi,k}$ with all invariant subspaces of H. From these intersections we select a nonzero subspace of minimal dimension, which we denote by $H'_{\varphi,k}$. We take all the invariant subspaces that intersect $H_{\varphi,k}$ in $H'_{\varphi,k}$. Among them is a minimal subspace H_1, the intersection of all these spaces.

Let us show that H_1 is irreducible. For suppose that H_1 can be decomposed into the direct sum of invariant subspaces

$$H_1 = H_{11} \dotplus H_{12}.$$

From the definition it follows that $H'_{\varphi,k}$ must be contained entirely in one of the subspaces H_{11} and H_{12}. But this contradicts the fact that H_1 is the minimal invariant subspace containing $H'_{\varphi,k}$.

So we have selected an invariant irreducible subspace H_1 of H. We consider the decomposition $H = H_1 \dotplus H_1'$, where H_1' is the orthogonal complement to H_1. Now H_1' is invariant. Consequently, there exists a space $H_{\varphi,k}$ with which H_1' has a nonzero intersection. Repeating the preceding arguments for H_1' we can select in H_1' an invariant irreducible subspace H_2, etc.

Continuing this process transfinitely, we obtain the required decomposition

$$H = \sum H_k \tag{4}$$

of H into a direct sum of invariant irreducible subspaces H_k. Since H is a separable space, the set of terms in this sum cannot be more than countable.

Finally, let us show that the multiplicity of each irreducible representation occurring in H is finite. We consider an arbitrary irreducible subspace H_k of H in (4). We can pick out an operator T_φ having in H_k an eigenvector with the eigenvalue $\lambda \neq 0$. But then clearly every space H_l in which an equivalent representation acts also contains an eigenvector of T_φ with the same eigenvalue $\lambda \neq 0$. However, there are only finitely many linearly independent eigenvectors of T_φ with the given eigenvalue $\lambda \neq 0$. Hence the

number of spaces H_l equivalent to H_k is finite. This completes the proof of the lemma.

So we have shown that the spectrum of the representation of a locally compact group G connected with a discrete subgroup Γ is discrete and of finite multiplicity, when $X = \Gamma \setminus G$ is a compact space.

We call a unitary representation T of a group G completely continuous if for every summable function φ on G the operator T_φ is completely continuous. We have shown that every completely continuous representation has a discrete spectrum of finite multiplicity. We can give a necessary and sufficient condition for T to be a completely continuous representation, in terms of its spectrum: the representation $T = \sum T_i$ is completely continuous if and only if:

1. Each irreducible component T_i is completely continuous.
2. The sequence $\{T_i\}$ has no limit points in the set $\hat{G}$ of all irreducible representations of G, endowed with the natural topology.

For a wide class of Lie groups, in particular for all semisimple and all nilpotent groups, it can be shown that every continuous unitary representation is completely reducible. Hence for these groups the first condition always holds. The second condition is clearly a sharper form of the postulate that the spectrum is of finite multiplicity.

4. The Trace Formula. Again, let $T(g)$ be a representation of a locally compact group G and induced by a discrete subgroup Γ such that $X = \Gamma \setminus G$ is a compact space. In **§ 2.3** we have shown that $T(g)$ has a discrete spectrum of finite multiplicity. In a certain sense, the trace formula, which we derive in this subsection, embraces a complete classification of all irreducible representations occurring in $T(g)$.

Since our operators $T(g)$, being unitary operators in an infinite-dimensional space, do not have a trace in the usual sense, we discuss the operators

$$T_\varphi = \int \varphi(g) T(g) \, dg.$$

As we have shown in **§ 2.2**, they are, under certain conditions on $\varphi(g)$, completely continuous integral operators. This is so, for example, when $\varphi(g)$ is a finite continuous function, and also in the more general case when $\varphi(g)$ satisfies the estimate

$$|\varphi(g_0)| \leq \int_U \varphi_1(g_0 g) \, dg, \tag{1}$$

where U is a compact neighborhood of the unit element of G, and $\varphi_1(g)$ is a nonnegative summable function on G.

We assume further that the T_φ are self-adjoint positive definite operators. This is true for functions of the form

$$\varphi(g) = \psi(g) * \overline{\psi(g^{-1})}.$$

The kernels $K(g_1, g_2)$ of the T_φ are given by the following formula:

$$K(g_1, g_2) = \sum_{\gamma\in\Gamma} \varphi(g_1^{-1}\gamma g_2)\chi(\gamma). \tag{2}$$

Here $\chi(\gamma)$ is a fixed finite-dimensional unitary representation of the subgroup Γ by which $T(g)$ was defined.

From the continuity of $K(g_1, g_2)$ and the compactness of $X = \Gamma \setminus G$ it follows that the self-adjoint positive definite operator T_φ has a trace, namely

$$\int_F \mathrm{Tr}\, K(g, g)\, dg = \int_F \Big(\sum_{\gamma\in\Gamma} \varphi(g^{-1}\gamma g)\ \mathrm{Tr}\ \chi(\gamma)\Big)\, dg, \tag{3}$$

where $\mathrm{Tr}\, K(g, g)$ is the trace of the matrix $K(g, g)$, $\mathrm{Tr}\ \chi(\gamma)$ is the trace of the matrix $\chi(\gamma)$, and F is a fundamental domain.

Now we calculate the trace of T_φ by another method. For this purpose we make a preliminary important assumption on the group G itself.

We assume that for every irreducible unitary representation $T_k(g)$ of G and every function $\varphi(g) \in S$, where S is a linear space of functions, everywhere dense in the space of continuous functions on G, the operator

$$T_\varphi^k = \int \varphi(g) T_k(g)\, dg \tag{4}$$

is completely continuous, has the trace $\mathrm{Tr}\,(T_\varphi^k)$; this trace $\mathrm{Tr}\,(T_\varphi^k)$ is a continuous function in S.

If this assumption is satisfied, then we can write

$$\mathrm{Tr}\,(T_\varphi^k) = \int \varphi(g)\sigma_k(g)\, dg, \tag{5}$$

where $\sigma_k(g)$ is a generalized function on G. It is natural to treat this generalized function $\sigma_k(g)$ as the trace of the representation operator $T_k(g)$ itself. It is usually called the *character* of the given representation $T_k(g)$.

The condition we have stated holds for all semisimple Lie groups.†

The importance of the concept of the character of a representation is clear from the following result: If G is a semisimple Lie group, then every irreducible unitary representation of it is uniquely determined by its character.

In later parts of the book we obtain explicit formulae for the characters of the irreducible representations of some groups.

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† In the case of semisimple Lie groups we can take for S the space of all finite infinitely differentiable functions on G.

Suppose then that G satisfies the assumption we have made. Let $H_1, \ldots, H_k, \ldots$ be the irreducible inequivalent subspaces into which the representation space of $T(g)$ splits, and $N_1, \ldots, N_k, \ldots$ the multiplicities with which they occur in this decomposition.

By hypothesis, T_φ has a trace in H_k, namely

$$\int \varphi(g)\sigma_k(g)\,dg,$$

where $\sigma_k(g)$ is the character of an irreducible representation in H_k. But then the trace of T_φ on the whole representation space of $T(g)$ is

$$\sum_{k=1}^{\infty} N_k \int \varphi(g)\sigma_k(g)\,dg. \tag{6}$$

When we equate the expressions (3) and (6) for the trace of T_φ, we obtain the required trace formula: *Let Γ be a discrete subgroup of G such that $X = \Gamma \setminus G$ is a compact space; let $T(g)$ be the representation of G induced by a finite-dimensional unitary representation $\chi(\gamma)$ of G. Let $\sigma_k(g)$ be the characters of the irreducible representations contained in $T(g)$, and N_k the multiplicities with which they occur in this representation. Then:*

$$\int_F \left(\sum_{\gamma\in\Gamma} \varphi(g^{-1}\gamma g)\ \mathrm{Tr}\ \chi(\gamma)\right) dg = \sum_{k=1}^{\infty} N_k \int_G \varphi(g)\sigma_k(g)\,dg. \tag{7}$$

In this formula $\mathrm{Tr}\ \chi(\gamma)$ denotes the trace of the matrix $\chi(\gamma)$.

Note that in the simplest case, when $\chi(\gamma)$ is the unit representation, the trace formula takes the following form:

$$\int_F \left(\sum_{\gamma\in\Gamma} \varphi(g^{-1}\gamma g)\right) dg = \sum_{k=1}^{\infty} N_k \int \varphi(g)\sigma_k(g)\,dg. \tag{8}$$

Let us discuss some consequences of the trace formula.

1. Let G be a compact group. Then we derive from the trace formula explicit expressions for the multiplicities N_k. For we set $\varphi(g) = \overline{\sigma_k(g)}$. It is well known that the characters of inequivalent representations are orthogonal to each other, that is,†

$$\int_G \overline{\sigma_k(g)}\,\sigma_m(g)\,dg = \begin{cases} 1, & \text{when } k = m, \\ 0, & \text{when } k \neq m. \end{cases}$$

Consequently the right-hand side of (7) gives us the multiplicity N_k with which the representation H_k occurs. But the left-hand side is equal to

$$\int_F \left(\sum_\gamma \overline{\sigma_k(g^{-1}\gamma g)}\ \mathrm{Tr}\ \chi(\gamma)\right) dg$$
$$= \sum_\gamma \overline{\sigma_k(\gamma)}\ \mathrm{Tr}\ \chi(\gamma) \int_F dg = \frac{1}{n_\Gamma} \sum_\gamma \overline{\sigma_k(\gamma)}\ \mathrm{Tr}\ \chi(\gamma).$$

† The measure dg is assumed to be normalized by the condition $\int dg = 1$.

(The volume of F is obviously equal to $1/n_\Gamma$, where n_Γ is of the order of Γ.)

So we find the following formula for the multiplicity N_k:

$$N_k = \frac{1}{n_\Gamma} \sum_{\gamma \in \Gamma} \overline{\sigma_k(\gamma)} \operatorname{Tr} \chi(\gamma). \tag{9}$$

In the particular case when $\chi(\gamma)$ is the unitary representation, this formula simplifies:

$$N_k = \frac{1}{n_\Gamma} \sum \sigma_k(\gamma). \tag{10}$$

2. Let G be the group of real matrices of order 2 with determinant 1. For the purpose of obtaining a similar device to determine the multiplicity N_k with which an irreducible representation H_k occurs in H, it is sufficient to find a function $\varphi(g)$ for which the estimate (1) holds, and in addition

$$\int_G \varphi(g)\sigma_m(g)\, dg = \begin{cases} 1, & \text{when} \quad m = k, \\ 0, & \text{when} \quad m \neq k. \end{cases} \tag{11}$$

Since the character $\sigma_m(g)$ depends only on the eigenvalues of the matrix $g \in G$, such a function $\varphi(g)$ is not necessarily unique.

Later we shall see that G has two types of representation—representations of the continuous series and representations of the discrete series.

It can be shown that for representations of the continuous series functions $\varphi(g)$ satisfying (11) do not exist.

For if $\varphi(g)$ satisfies (11), then in any case

$$\int_G |\varphi(g)|\, dg < \infty.$$

We consider the function

$$h(s) = \int_G \varphi(g)\sigma_s(g)\, dg,$$

where $\sigma_s(g)$ is the character of the representation of the principal continuous series with index s (s is a purely imaginary number). This character is given by the following formula:

$$\sigma_s(g) = \frac{|\lambda_g|^{-s} + |\lambda_g|^{s}}{|\lambda_g - \lambda_g^{-1}|},$$

in case the eigenvalues λ_g and λ_g^{-1} of g are real; and

$$\sigma_s(g) = 0,$$

in case λ_g and λ_g^{-1} are complex.

It can be shown that the function $h(s)$ is analytic in the domain $|\operatorname{Re} s| < 1$. For our purposes we need a function $\varphi(g)$ for which $h(s)$ differs from zero only near a fixed point s_0. Clearly such functions $\varphi(g)$ do not exist.

For the representations of the discrete series the position is different: for them such a function does exist.

For let $T_n(g)$ be a representation of the discrete series with the index n, realized in the space of functions that are analytic in the upper half-plane (see

below **§ 3.3**). Then we may take for the required function $\varphi(g)$

$$\varphi_n(g) = \frac{e^{in\theta}(\mathrm{Im}\ z)^{n/2}}{(z+i)^n},$$

where $z = -\dfrac{g_{22}+ig_{21}}{g_{12}+ig_{11}}$, $\theta = \arg(g_{12}-ig_{11})$. Substituting $\varphi_n(g)$ in (7) we obtain a finite expression for the multiplicity N_n with which $T_n(g)$ occurs in $T(g)$. We shall find this later, in § **5.7**, by applying somewhat different methods.

5. Another Form of the Trace Formula. Let us transform the trace formula into a more convenient form. We shall show that the left-hand side of the equation (7) in **§ 2.4** can be brought to the following form:

$$\sum_{\gamma}{}' \mu(\Gamma_\gamma \setminus G_\gamma)\ \mathrm{Tr}\ \chi(\gamma) \int_{G_\gamma\backslash G} \varphi(g^{-1}\gamma g)\ dg, \tag{1}$$

where γ ranges over exactly one representation from each class of conjugate elements in Γ. Here G_γ denotes the centralizer of γ in G; thus,

$$I_\gamma = \int_{G_\gamma\backslash G} \varphi(g^{-1}\gamma g)\ dg$$

is the integral of φ over the class of all elements in G that are conjugate to γ; Γ_γ denotes the centralizer of γ in Γ; $\mu(\Gamma_\gamma \setminus G_\gamma)$ is the measure of the space $\Gamma_\gamma \setminus G_\gamma$ (it is not difficult to check that this measure is finite).

To prove (1) is a valid formula, we split the elements γ into conjugacy classes with respect to Γ. Clearly the expression Tr $\chi(\gamma)$ is constant on each of these classes.

Therefore the left-hand side of the trace formula can be written:

$$\sum \mathrm{Tr}\ \chi(\gamma) \int_F \Big(\sum_{\gamma'} \varphi(g^{-1}\gamma' g)\Big) dg,$$

where the inner summation is over a class of conjugate elements and the outer one over the set of these classes. Now we transform the expression under the inner sum sign.

We select one of the conjugacy classes. It consists of elements of the form

$$\gamma' = \gamma_i^{-1}\gamma\gamma_i,$$

where γ is fixed and γ_i ranges over Γ. Note that when γ_i ranges over Γ, each such element γ' is obtained several times. For two elements γ_i and γ_j determine one and the same element γ' if and only if $(\gamma_i\gamma_j^{-1})\gamma(\gamma_i\gamma_j^{-1}) = \gamma$, that is, if $\gamma_i\gamma_j^{-1}$ belongs to the centralizer Γ_γ of γ in Γ. Hence, to obtain each element γ' of a conjugacy class precisely once, the γ_i must range over only one representative of

each coset $\Gamma_\gamma \setminus \Gamma$. So we have

$$\sum_{\gamma'} \varphi(g^{-1}\gamma' g) = \sum_{\gamma_i \in \Gamma_\gamma \setminus \Gamma} \varphi(g^{-1}\gamma_i^{-1}\gamma\gamma_i g),$$

where γ_i ranges over a set of coset representatives of $\Gamma_\gamma \setminus \Gamma$; consequently,

$$\int_F \Big(\sum_{\gamma'} \varphi(g^{-1}\gamma' g)\Big)\, dg = \int_F \Big(\sum_{\gamma_i} \varphi(g^{-1}\gamma_i^{-1}\gamma\gamma_i g)\Big)\, dg$$
$$= \int_{F_1} \varphi(g^{-1}\gamma g)\, dg,$$

where the integral is taken over the set

$$F_1 = \sum_{\gamma_i \in \Gamma_\gamma \setminus \Gamma} \gamma_i F.$$

It is clear that F_1 is a fundamental set of the subgroup Γ_γ in G. So we have

$$\int_F \Big(\sum_{\gamma'} \varphi(g^{-1}\gamma' g)\Big)\, dg = \int_{\Gamma_\gamma \setminus G} \varphi(g^{-1}\gamma g)\, dg.$$

Now let G_γ be the centralizer of γ in the whole group G. Then we have

$$\int_{\Gamma_\gamma \setminus G} \varphi(g^{-1}\gamma g)\, dg = \int_{G_\gamma \setminus G} \int_{\Gamma_\gamma \setminus G_\gamma} \varphi(g^{-1}g_1^{-1}\gamma g_1 g)\, dg_1\, dg$$
$$= \int_{G_\gamma \setminus G} \int_{\Gamma_\gamma \setminus G_\gamma} \varphi(g^{-1}\gamma g)\, dg_1\, dg$$
$$= \mu(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma \setminus G} \varphi(g^{-1}\gamma g)\, dg.$$

Hence, when γ' ranges over the set of conjugates to γ in Γ, we have

$$\int_F \Big(\sum_{\gamma'} \varphi(g^{-1}\gamma' g)\Big)\, dg = \mu(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma \setminus G} \varphi(g^{-1}\gamma g)\, dg.$$

Multiplying this equation by $\operatorname{Tr} \chi(\gamma)$ and summing over the set of conjugacy classes in Γ we obtain the required expression (1).

We state the final result. *Let G be a locally compact group and Γ a discrete subgroup for which $\Gamma \setminus G$ is a compact space; let $\chi(\gamma)$ be a unitary representation of Γ and $T(g)$ the representation of G induced by $\chi(\gamma)$.*

Let $\sigma_k(g)$, $k = 1, 2, \ldots,$ be the characters of the irreducible unitary

representations contained in $T(g)$ and N_k their multiplicities. Then for every finite function $\varphi(g)$ on G the following trace formula holds:

$$\sum_k N_k \int \varphi(g)\sigma_k(g)\, dg = \sum_\gamma \operatorname{Tr} \chi(\gamma)\mu(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma \setminus G} \varphi(g^{-1}\gamma g)\, dg. \quad (2)$$

Here $\operatorname{Tr} \chi(\gamma)$ is the trace of the matrix $\chi(\gamma)$; G_γ and Γ_γ are the centralizers of γ in G and Γ, respectively; the summation is taken over the set of conjugacy classes in Γ. We note that

$$\int_{G_\gamma \setminus G} \varphi(g^{-1}\gamma g)\, dg$$

is nothing but the integral over the class of elements conjugate to γ in G.

The trace formula can be presented in yet another form, by going over from $\varphi(g)$ to its Fourier transform. For let $\sigma(g)$ range over the characters of all irreducible unitary representations of G. From the function $\varphi(g)$ on G we go over to its Fourier transform

$$h(\sigma) = \int \varphi(g)\sigma(g)\, dg. \quad (3)$$

Then the left-hand side of the trace formula can be written in the form

$$\sum_k h(\sigma_k),$$

where the sum is taken over all the irreducible representations σ_k contained in $T(g)$, and each term is repeated as often as the multiplicity of the representation indicates.

On the other hand, every term on the right-hand side of the trace formula can also be expressed by $h(\sigma)$ as a certain integral

$$\int h(\sigma)\psi_\gamma(\sigma)\, d\sigma.$$

The integral is taken over the set of all irreducible unitary representations of G. Then the trace formula assumes the following form:

$$\sum_k h(\sigma_k) = \sum_\gamma \int h(\sigma)\psi_\gamma(\sigma)\, d\sigma,$$

where the summation on the right is taken over the set of conjugacy classes in Γ.

Our problem is to obtain an explicit expression for the function $\psi_\gamma(\sigma)$. In **§ 5** this problem will be solved for the group of real unimodular matrices of order 2.

§ 3. IRREDUCIBLE UNITARY REPRESENTATIONS OF THE GROUP OF REAL UNIMODULAR MATRICES OF ORDER 2

In this section we give a classification of the irreducible unitary representations of the group G of real unimodular matrices of order 2. As a rule, the results are stated without proof. The reader will find details on the representations of G in Chapter 2, where we study representations of the group of matrices of order 2 with elements from an arbitrary locally compact field (see also Gel'fand et al. [27], Chapter 7).

1. The Principal Series of Irreducible Unitary Representations. The irreducible unitary representations of G, other than the unit representation, split into three series—the principal continuous, the supplementary, and the discrete. To begin with we describe the simplest class of irreducible unitary representations—the representations of the principal series.

We consider an affine plane X *with the origin of coordinates deleted* (henceforth, when we talk of an affine plane X, we always assume that the origin of coordinates is deleted and that the appropriate topology is introduced in X). The plane X is a homogeneous space in which G acts as a group of affine transformations. For the element $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ of G carries an arbitrary point $x = (x_1, x_2)$ into

$$xg = (x_{11}g_{11} + x_2g_{21}, x_1g_{12} + x_2g_{22}). \tag{1}$$

We consider the space H of measurable functions $f(x)$, $x \in X$, of integrable square:

$$\int |f(x)|^2 \, dx \equiv \int |f(x_1, x_2)|^2 \, dx_1 \, dx_2 < \infty.$$

We give the representation operators $T(g)$ by the following formula:

$$T(g)f(x) = f(xg). \tag{2}$$

This representation is unitary, because the measure $dx \equiv dx_1 \, dx_2$ is preserved under the transformations $x \to xg$.

The representation $T(g)$ is reducible. By decomposing it into irreducible representations we obtain the principal series of representations. Let us show how this decomposition can be carried out.

First of all we divide the set of functions $f(x)$ into even and odd ones. The even and odd functions obviously form invariant subspaces which we denote, respectively, by H^+ and H^-.

Consider, for example, the space of even functions $f^+(x)$. We set

$$f_s^+(x) = \int_0^\infty f^+(tx)t^{-s}\,dt, \tag{3}$$

where s is pure imaginary. The functions $f_s^+(x)$ are homogeneous; indeed,

$$f_s^+(tx) = |t|^{s-1} f_s^+(x). \tag{4}$$

We introduce a norm in the space H_s^+ of functions $f_s^+(x)$ by setting

$$\|f_s^+\|^2 = \int_{|x|=1} |f_s^+(x)|^2\,d\varphi, \tag{5}$$

where $d\varphi$ is the Haar measure on the unit circle $|x| = 1$. It is easy to verify that the representation $T(g)$ in H_s^+ given by the formula

$$T(g)f_s^+(x) = f_s^+(xg) \tag{6}$$

is unitary. This representation is irreducible. Similarly we construct representations in the spaces H_s^- of odd functions.

The representations in the spaces H_s^+ and H_s^- form the so-called principal continuous series of irreducible unitary representations of G. In what follows it is convenient to call the representations in the spaces H_s^+ of even functions representations of the first principal series, and those in the spaces H_s^- of odd functions representations of the second principal series.

It can be shown that the representations corresponding to s and $-s$ (in the case of even and of odd functions separately) are equivalent and that otherwise the representations are inequivalent; thus, each irreducible representation occurs here twice. This point is very important for us later.

The representation in H splits into these representations of the principal series. For from the Parseval formula for the Mellin transform it is immediately clear that

$$\|f\|^2 = \int_{-\infty}^{+\infty} \|f_s^+\|^2\,d(is) + \int_{-\infty}^{+\infty} \|f_s^-\|^2\,d(is). \tag{7}$$

There is another realization of the representations of the principal continuous series.

We obtain it by taking instead of the homogeneous functions of two variables $f(x_1, x_2)$ functions of a single variable $\varphi(x)$,

connected with the f by the following relation:

$$\varphi(x) = f(x, 1). \tag{8}$$

(Obviously the homogeneous function $f_s(x_1, x_2)$ is uniquely determined by $\varphi(x)$.) Then we obtain the following realization of the representations of the principal continuous series.

The representations of the first principal series H_s^+ are realized in the space of functions $\varphi(x)$ on the real line with the scalar product

$$(\varphi_1, \varphi_2) = \int_{-\infty}^{+\infty} \varphi_1(x)\overline{\varphi_2(x)}\, dx. \tag{9}$$

The representation operators are given by the following formula:

$$T(g)\varphi(x) = \varphi\left(\frac{g_{11}x + g_{21}}{g_{12}x + g_{22}}\right)|g_{12}x + g_{22}|^{s-1}, \tag{10}$$

where s is imaginary.

The representations of the second principal series H_s^- are also realized in the space of functions on the real line with the scalar product (9). But the representation operators are given by the following formula:

$$T(g)\varphi(x) = \varphi\left(\frac{g_{11}x + g_{21}}{g_{12}x + g_{22}}\right)|g_{12}x + g_{22}|^{s-1} \operatorname{sign}(g_{12}x + g_{22}). \tag{11}$$

2. The Supplementary Series of Representations. Let $s \neq 0$ be a real number in the interval $-1 < s < 1$. By H_s^+ we denote the space of even functions $f_s^+(x)$, satisfying the condition of homogeneity (4). The scalar product in H_s^+ is given by the following formula:

$$(f_s', f_s'') = \int_{|x'|=1} \int_{|x''|=1} K_s(x', x'')f_s'(x')\overline{f_s''(x'')}\, d\varphi'\, d\varphi'', \tag{1}$$

where

$$K_s(x', x'') = |x_1'x_2'' - x_2'x_1''|^{-s-1}. \tag{2}$$

(Where $s < 0$, the integral must be understood in the sense of the regularizing value [27].)

We define the representation operator $T(g)$, as before, by the formula (6) of **§ 3.1**. It can be shown that the representations $T(g)$ are unitary and irreducible; they are equivalent only for s and $-s$. These representations are said to belong to the *supplementary series*.

We now indicate another realization of the representations of the supplementary series. It is obtained when instead of the homogeneous functions of two variables $f(x_1, x_2)$ we consider functions of a single variable $\varphi(x) = f(x, 1)$.

A representation of the supplementary series H_s^+, where $s \neq 0$ is a real number in the interval $-1 < s < 1$, is realized in the space of functions on the real line with the scalar product

$$(\varphi_1, \varphi_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x_1 - x_2|^{-s-1} \varphi_1(x_1) \overline{\varphi_2(x_2)} \, dx_1 \, dx_2. \tag{3}$$

The representation operators are given by the formula

$$T(g) \varphi(x) = \varphi\left(\frac{g_{11}x + g_{21}}{g_{12}x + g_{22}}\right) |g_{12}x + g_{22}|^{s-1}. \tag{4}$$

3. The Discrete Series of Representations. This series consists of two parts. One half is realized in the space of functions of a complex variable z that are analytic in the upper half-plane $\operatorname{Im} z > 0$. The representation operators are given by the following formula:

$$T(g) \varphi(z) = \varphi\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right) (g_{12}z + g_{22})^{-n-1}, \tag{1}$$

where n is a nonnegative integer which determines the representation. The scalar product is defined as follows:

$$(\varphi_1, \varphi_2) = \int_{\operatorname{Im} z \geq 0} \varphi_1(z) \overline{\varphi_2(x)} y^{n-1} \, dx \, dy, \qquad \text{when} \quad n > 0, \tag{2}$$

$$(\varphi_1, \varphi_2) = \int_{-\infty}^{+\infty} \varphi_1(x) \overline{\varphi_2(x)} \, dx, \qquad \text{when} \quad n = 0, \tag{3}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are the boundary values of the analytic functions $\varphi_1(z)$ and $\varphi_2(z)$ on the real axis.

The other half of the discrete series is realized in the space of functions that are analytic in the lower half-plane. The representation operators are given by the same formula (1) as in the case of the first half of the discrete series. The scalar product is defined by a formula analogous to (2). No two representations of the discrete series are equivalent.

4. Another Realization of the Representations of the Principal and the Supplementary Series. Here we discuss two classes of representations: representations of the first principal series realized in the space H_s^+ of even functions, and representations of the supplementary series.

These representations have the following important property. The representation space contains a vector $f_0(x)$ invariant under the

operators $T(u)$, where u ranges over the orthogonal matrices:

$$u = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Indeed, such a vector is

$$f_0(x) = (x_1^2 + x_2^2)^{\frac{s-1}{2}}. \tag{1}$$

Let us show that there are no other vectors invariant under the operators $T(u)$.

Let f be a vector for which

$$T(u)f = f,$$

that is

$$f(x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t) = f(x_1, x_2)$$

for every t. From this equation it follows immediately that f is a function of $x_1^2 + x_2^2$ only: $f = f(x_1^2 + x_2^2)$. But since furthermore f is homogeneous of degree $s - 1$, it is clear that $f = C(x_1^2 + x_2^2)^{\frac{s-1}{2}}$.

Using this property of the representations we now construct another realization in the space of functions on $U \setminus G$, where U is the subgroup of orthogonal matrices.

With every function $f(x)$ in the representation space H_s^+ we associate the function $\varphi(g)$ on G defined by the following formula:

$$\varphi(g) = (T(g)\mathrm{x}, f_0). \tag{2}$$

Here f_0 is a vector in H_s^+ invariant under the operators $T(u)$, and the expression on the right denotes the scalar product in H_s^+.

We show that *the map*

$$f(x) \to \varphi(g)$$

is one to one.

For the kernel of this map is obviously an invariant subspace of H_s^+. Since H_s^+ is irreducible, this kernel either coincides with H_s^+ or is the null space. But it cannot coincide with H_s^+, because the image of the function $f_0(x)$ is different from zero. Therefore the kernel of the map $f \to \varphi$ is the null space.

From the functions $f(x)$ we go over to the functions

$$\varphi(g) = (T(g)f, f_0)$$

and in this way we interpret the representation in the space of the functions $\varphi(g)$. Let us show that in the space of functions $\varphi(g)$ the representation operator $T(g)$ is given by the following formula:

$$T(g_0)\varphi(g) = \varphi(gg_0). \tag{3}$$

For if we apply to f the operator $T(g_0)$, then the corresponding function $\varphi(g) = (T(g)f, f_0)$ goes over into

$$\varphi_1(g) = (T(g)T(g_0)f, f_0) = (T(gg_0)f, f_0) = \varphi(gg_0).$$

As a result we obtain a new realization of the representation $T(g)$, which is constructed in a certain space of functions $\varphi(g)$ on G. The representation operator $T(g)$ is given by (3).

We denote the space of functions $\varphi(g)$ so obtained, as before, by H_s^+.

Let us study the main properties of the functions $\varphi(g)$ in the space H_s^+:

1. The functions $\varphi(g)$ are bounded. For from the equation $\varphi(g) = (T(g)f, f_0)$ it follows that $|\varphi(g)|^2 \leq \|T(g)f\| \; \|f_0\|$. But $T(g)$ is a unitary operator, therefore $\|T(g)f\| = \|f\|$. So we have $|\varphi(g)|^2 \leq \|f\| \; \|f_0\|$, that is, $\varphi(g)$ is a bounded function.

2. The functions $\varphi(g)$ are infinitely differentiable in g. To prove this we write down an explicit expression of $\varphi(g)$ in terms of the function $f(x_1, x_2)$. By definition

$$\varphi(g) = (f, T(g^{-1})f_0).$$

Using the formula for the scalar product in H_s^+ (see **§ 3.1** and **§ 3.2**) and the formula for the operator $T(g)$ we may write this expression in the following explicit forms:

For a representation of the principal series

$$\varphi(g) = \int\limits_{x_1^2+x_2^2=1} f(x_1, x_2)[(x_1g_{22} - x_2g_{21})^2 + (-x_1g_{12} + x_2g_{11})^2]^{\frac{s-1}{2}} \, d\varphi. \tag{4}$$

For a representation of the supplementary series

$$\varphi(g) = \int\limits_{\substack{x_1^2+x_2^2=1 \\ y_1^2+y_2^2=1}} f(x_1, x_2)|x_1y_2 - x_2y_1|^{-s-1} \, [(y_1g_{22} - y_2g_{21})^2 + (-y_1g_{12} + y_2g_{11})^2]^{\frac{s-1}{2}} \, d\varphi_1 \, d\varphi_2. \tag{5}$$

The fact that $\varphi(g)$ is an infinitely differentiable function of the variables g_{ij} follows immediately from these formulae. Observe that if L_g is an arbitrary linear differential operator on G, then the formula for differentiation under the integral sign holds:

$$L_g\varphi(g) = (f, L_gT(g^{-1})f_0).$$

The detailed verification of these facts is left to the reader.

3. The functions $\varphi(g)$ are constant on the right cosets in G of the subgroup U of orthogonal matrices, that is,

$$\varphi(ug) = \varphi(g)$$

for every orthogonal matrix

$$u = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

For we have

$$\varphi(ug) = (T(ug)f, f_0) = (T(u)T(g)f, f_0) = (T(g)f, T(u^{-1})f_0).$$

But the vector f_0 is invariant under $T(u)$, that is, $T(u)f_0 = f_0$. Consequently

$$\varphi(ug) = (T(g)f, f_0) = \varphi(g).$$

By what we have proved, the functions $\varphi(g)$ may be regarded as given in the space of cosets $U \setminus G$. Let us show that the homogeneous space $U \setminus G$ is isomorphic to the half-plane $\operatorname{Im} z > 0$ of the plane of the complex variable z on which G acts as a group of linear fractional transformations.

On the half-plane $\operatorname{Im} z > 0$ we fix the point $z_0 = i$ and associate with every point z the set of all elements g of G that carry i into this point z, that is,

$$\frac{ig_{11} + g_{21}}{ig_{12} + g_{22}} = z.$$

This defines a map of G onto the upper half-plane. The set of elements g that are carried under this map into i, that is, for which

$$\frac{ig_{11} + g_{21}}{ig_{12} + g_{22}} = i,$$

obviously forms the subgroup U of orthogonal matrices. Hence it is clear that under this map the inverse images of the points z are the cosets Ug. As a result we obtain a one-to-one correspondence between the cosets Ug and the points z of the half-plane $\operatorname{Im} z > 0$. Clearly under the transformation $g \to gg_0$ the corresponding point z is subjected to the fractional-linear transformation with the matrix g_0:

$$z \to z' = \frac{zg_{11} + g_{21}}{zg_{12} + g_{22}}.$$

So the functions φ can be regarded as functions on the upper half-plane $\operatorname{Im} z > 0$, and we write $\varphi(z)$ instead of $\varphi(g)$.

Let us find the expression of a function $\varphi(z)$ in terms of the original functions $f(x_1, x_2)$. Let $z = x + iy$ be a point of the upper half-plane, and Ug the coset that is the inverse image of z; we recall that this set consists of the matrices that carry i into z. It is easily checked that we can take as representative of this class the following matrix:

$$g_z = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ xy^{-\frac{1}{2}} & y^{-\frac{1}{2}} \end{pmatrix}. \tag{6}$$

So the functions $\varphi(z)$ are expressed in terms of the original functions $f(x_1, x_2)$ by the following formula:

$$\varphi(z) = (T(g_z)f, f_0).$$

where g_z is a matrix of the form (6).

We state the final result.

The space H_s^+ of an irreducible representation of the first principal or the supplementary series may be realized as a space of infinitely differentiable bounded functions $\varphi(z)$ on the upper half plane $\operatorname{Im} z > 0$. In this realization the representation operator $T(g)$ is defined by the following formula:

$$T(g)\,\varphi(z) = \varphi\left(\frac{zg_{11} + g_{21}}{zg_{12} + g_{22}}\right).$$

Observe that we have not obtained a complete classification of the functions $\varphi(z)$ of which the space H_s^+ consists. Furthermore, we have not found an explicit formula for the scalar product in the space of functions $\varphi(z)$.

In the next subsection we obtain additional information on the space H_s^+.

5. The Laplace Operator Δ. The Space Ω_s. We examine the space of all infinitely differentiable functions $\varphi(z)$ on the upper half-plane $\operatorname{Im} z > 0$. We define representation operators $T(g)$ of the group G by the following formula:

$$T(g)\,\varphi(z) = \varphi\left(\frac{zg_{11} + g_{21}}{zg_{12} + g_{22}}\right). \tag{1}$$

We show that on the half-plane $\operatorname{Im} z > 0$ there exists a differential operator of the second order Δ that is permutable with all the operators $T(g)$:

$$\Delta T(g) = T(g)\Delta \tag{2}$$

or in more detail,

$$\Delta\left[\varphi\left(\frac{zg_{11} + g_{21}}{zg_{12} + g_{22}}\right)\right] = (\Delta\varphi)\left(\frac{zg_{11} + g_{21}}{zg_{12} + g_{22}}\right). \tag{2'}$$

For let us consider the operator

$$\Delta = (z - \bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{3}$$

We show that it satisfies the relation (2). We set $z' = \dfrac{zg_{11} + g_{21}}{zg_{12} + g_{22}}$. Then we have: $\dfrac{\partial}{\partial z'} = (zg_{12} + g_{22})^2 \dfrac{\partial}{\partial z}$, $\dfrac{\partial}{\partial \bar{z}'} = (\bar{z}g_{12} + g_{22})^2 \dfrac{\partial}{\partial \bar{z}}$, $z' - \bar{z}' = |zg_{12} + g_{22}|^{-2}\,(z - \bar{z})'$ Hence we immediately obtain

$$(z' - \bar{z}')^2 \frac{\partial^2}{\partial z'\,\partial \bar{z}'} = (z - \bar{z})\,\frac{\partial^2}{\partial z\,\partial \bar{z}}.$$

Obviously, this equation is equivalent to (2).

So we have shown that the operator $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ commutes with the operators $T(g)$. We call this Δ the *Laplace operator* on the half-plane $\operatorname{Im} z > 0$.

It is not difficult to verify that every other differential operator of the second order that commutes with the operators $T(g)$ is of the form

$$\alpha\Delta + \beta,$$

where α, β are constants. For suppose that the operator

$$\Delta_1 = a_1(z)\frac{\partial^2}{\partial z^2} + a_2(z)\frac{\partial^2}{\partial \bar{z}^2} + a_3(z)\frac{\partial^2}{\partial z\,\partial \bar{z}} + a_4(z)\frac{\partial}{\partial z} + a_5(z)\frac{\partial}{\partial \bar{z}} + a_6(z)$$

satisfies the commutativity condition (2). When the matrix g has the form $g = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$, this condition yields: $a_i(z + \gamma) = a_i(z)$, where γ is any real number, $i = 1, \ldots, 6$. Consequently the coefficients $a_i(z)$ depend only on $\operatorname{Im} z = y$. So we have

$$a_i(z) = a_i(y).$$

Next we apply condition (2) to the case when $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. We find that $a_i(\lambda^2 y) = \lambda^4 a_i(y)$ for $i = 1, 2, 3$; $a_i(\lambda^2 y) = \lambda^2 a_i(y)$ for $i = 4, 5$; $a_6(\lambda^2 y) = a_6(y)$. Consequently $a_i(y) = \alpha_i y^2$ for $i = 1, 2, 3$; $a_i(y) = \alpha_i y$ for $i = 4, 5$; $a_6(y) = \alpha_6$ where the α_i are constant.

Finally, by applying condition (2) to the case when $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we easily check that $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = 0$.

In the preceding subsection we obtained a realization of the representations of the principal and supplementary series of G in certain subspaces H_s^+ of infinitely differentiable functions $\varphi(z)$, $\operatorname{Im} z > 0$. Let us clarify what form the Laplace operator Δ assumes in each of these subspaces.

We shall show that on H_s^+ the Laplace operator Δ is a multiple of the unit operator, namely:

$$\Delta = \frac{1 - s^2}{4} E, \tag{4}$$

where E is the unit operator.

For the proof we make use of the integral representation of the functions $\varphi(z)$:

$$\varphi(z) = (T(g_z)f, f_0) = (f, T(g_z^{-1})f_0),$$

where $f_0(x) = (x_1^2 + x_2^2)^{\frac{s-1}{2}}$, and the scalar product $(.\,,.)$ is defined, as in **§ 3.1** and **§ 3.2** by the following formulae:

$$(f_1, f_2) = \int_{|x|=1} f_1(x)\overline{f_2(x)}\, d\varphi$$

for the representations of the principal series;

$$(f_1, f_2) = \int\limits_{|x'|=1} \int\limits_{|x''|=1} |x_1'x_2'' - x_1''x_2'|^{-s-1} f_1(x') f_2(x'')\, d\varphi'\, d\varphi''$$

for the representations of the supplementary series. Differentiating under the integral sign we find that

$$\Delta\varphi(z) = (f, \Delta Y(g_z^{-1}) f_0).$$

Hence it is sufficient to verify that

$$\Delta T(g_z^{-1}) f_0(x) = \frac{1 - s^2}{4} T(g_z^{-1}) f_0(x). \tag{5}$$

The validity of the relation (5) comes out by a direct check. We have

$$g_z^{-1} = \begin{pmatrix} y^{-\frac{1}{2}} & 0 \\ -xy^{-\frac{1}{2}} & y^{\frac{1}{2}} \end{pmatrix}.$$

Consequently,

$$T(g_z^{-1}) f_0 = T(g_z^{-1})(x_1^2 + x_2^2)^{\frac{s-1}{2}} = [y^{-1}(x_1 - xx_2)^2 + yx_2^2]^{\frac{s-1}{2}}.$$

We rewrite this expression in the variables $z = x + iy$ and $\bar{z}$:

$$T(g_z^{-1}) f_0 = \left[2i \frac{(x_2 z - x_1)(x_2 \bar{z} - x_1)}{z - \bar{z}}\right]^{\frac{s-1}{2}}.$$

When we now apply the Laplace operator $\Delta = (z - \bar{z})^2 \dfrac{\partial^2}{\partial z\, \partial \bar{z}}$ to this expression, we obtain

$$\Delta \left[2i \frac{(x_2 z - x_1)(x_2 \bar{z} - x_1)}{z - \bar{z}}\right]^{\frac{s-1}{2}} = \frac{1 - s^2}{4} \left[2i \frac{(x_2 z - x_1)(x_2 \bar{z} - x_1)}{z - \bar{z}}\right]^{\frac{s-1}{2}}.$$

This proves (5).

We now give a final statement of the results obtained. Let $T(g)$ be a representation of the fundamental or the supplementary series, f_0 a vector in the representation space H_s^+ and invariant under the operators $T(u)$, where u ranges over the set of orthogonal matrices. With every vector f from the representation space we associate the function $\varphi(g)$ defined by the formula

$$\varphi(g) = (T(g) f, f_0).$$

In **§ 3.3** we have shown that the correspondence $f \to \varphi$ is one to one and that the functions $\varphi(g)$ are bounded and infinitely differentiable. Next we have established that the functions φ are constant on the cosets of the subgroup U of orthogonal matrices, so that they can be

treated as functions in the coset space $U \backslash G$. Since $U \backslash G$ is isomorphic to the upper half-plane $\operatorname{Im} z > 0$ of the plane of the complex variable z, φ may also be regarded as a function $\varphi(z)$ given on the upper half-plane. The original representation $T(g)$ can be assumed to be given in the space of these functions $\varphi(z)$. Then the representation operators $T(g)$ are given by the following formula:

$$T(g)\varphi(z) = \varphi\left(\frac{zg_{11} + g_{21}}{zg_{12} + g_{22}}\right). \tag{6}$$

In this subsection we have shown that the functions $\varphi(z)$ are eigenfunctions of the operator $\Delta = (z - \bar{z})^2 \dfrac{\partial^2}{\partial z \, \partial \bar{z}}$, corresponding to the eigenvalue $\dfrac{1 - s^2}{4}$:

$$\Delta\varphi(z) = \frac{1 - s^2}{4} \varphi(z). \tag{7}$$

Note that the converse statement does not hold:† not every solution of (7) belongs to the space H_s^+.

Occasionally it is convenient not to use the Hilbert space H_s^+, in which the representations were constructed in the first instance, but an extension of it—the space Ω_s of *all* solutions of the equation (7). We can endow this space Ω_s with a natural topology in which it becomes a complete topological space. Obviously, the representation $T(g)$ extends from H_s^+ to the whole space Ω_s; the representation operators are given on Ω_s by the same formula (6).

We call this space Ω_s the *complete space connected with the given irreducible representation* $T(g)$. It plays a fundamental role in the duality theorem (**§ 4**).

Without proof we state some properties of Ω_s:

1. H_s^+ is an everywhere dense subset of Ω_s.

2. Ω_s is irreducible in all reasonable interpretations of this term [27]. In particular, Ω_s does not contain a closed invariant subspace; there is no bounded operator in Ω_s, other than the unit operator, that commutes with all the operators $T(g)$.

§ 4. THE DUALITY THEOREM

In **§ 2** we raised the following problem: for a given representation $T(g)$ of the group G and induced by a discrete subgroup Γ,

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† For example, the equation (7) has unbounded solutions. However, all the functions $\varphi(z) \in H_s^+$ are bounded.

it is required to find the spectrum of $T(g)$, in other words, to decompose this representation into irreducible ones.

In this section we establish a connection between this problem and the classical problems of the theory of automorphic forms. We show that the multiplicity with which a given irreducible representation occurs in $T(g)$ is equal to the dimension of the space of automorphic forms associated with this irreducible representation. The concept of an automorphic form will be explained later.

The arguments to be used here are very easy to understand when G is a *compact* group and Γ any subgroup of it, not necessarily discrete.

Let $\chi(\gamma)$ be an irreducible representation of Γ and $T(g)$ the representation of G induced by it. We want to find the multiplicity with which a given irreducible representation $T_k(g)$ is contained in $T(g)$. For this purpose we consider the operators $T_k(\gamma)$, where γ ranges over Γ. They form a representation of Γ, which is, in general, reducible. Now it turns out that the following theorem holds.

DUALITY THEOREM. *The multiplicity with which an irreducible representation $T_k(g)$ is contained in $R(g)$ is equal to the multiplicity with which the irreducible representation $\chi(\gamma)$ of Γ is contained in $T_k(\gamma)$.*

We mention the special case of this theorem when $\chi(\gamma)$ is the unit representation of Γ. In this case $T(g)$ is the representation in the space of functions $f(x)$ on $X = \Gamma \setminus G$, defined by the formula

$$T(g)f(x) = f(xg).$$

The duality theorem can then be stated as follows. *The multiplicity with which the irreducible representation $T_k(g)$ is contained in $T(g)$ is equal to the number of linearly independent vectors ξ in the representation space $T_k(g)$ that are invariant under Γ, that is, for which*

$$T_k(\gamma)\,\xi = \xi$$

for every $\gamma \in \Gamma$.

Our task is to extend this result to noncompact groups. Note that the duality theorem does not directly go over to noncompact groups G. The main reason for this is the fact that, since irreducible unitary representations of such groups G are, in general, infinite-dimensional, invariant vectors under Γ need not lie in the Hilbert space of the representation.

Nevertheless, an analogue to the duality theorem can be obtained for every semisimple Lie group G and a discrete subgroup of it for which the space $X = \Gamma \setminus G$ is compact. For this purpose we have to extend the set of functions on which the operators of the irreducible representation $T_k(g)$ act. In other words, the irreducible representation $T_k(g)$ must be given not in the Hilbert space, but in a certain extension Ω_k of it. For a number of Lie groups this space Ω_k can be described effectively.

We begin with a detailed account of the duality theorem for the case of the group of real matrices of order 2. For simplicity we discuss here not all the representations induced by subgroups Γ, but only the simplest of them—the representation generated by the homogeneous space $X = \Gamma \setminus G$.

The general results concerning an arbitrary semisimple Lie group Γ are given at the end of the section.

1. Automorphic Forms. In this subsection we derive the duality theorem for the group G of real unimodular matrices of order 2. To state the theorem we need the concept of an automorphic form.

Let $T(g)$ be a unitary irreducible representation of a group G and let Γ be a discrete subgroup of G. Automorphic forms (relative to Γ) ought to be defined as vectors ξ that are invariant in the representation space under the operators $T(\gamma)$, $\gamma \in \Gamma$:

$$T(\gamma)\,\xi = \xi.$$

However, when the representation is infinite-dimensional, this definition is not entirely satisfactory; it is natural to ask that the vectors ξ should not necessarily lie in the representation space, but in a certain extension of it.

So we introduce a precise concept of an automorphic form corresponding to the given irreducible representation of G. We recall that G has three series of irreducible representations—the principal continuous, the supplementary and the discrete series.

We begin with representations of the discrete series. According to **§ 3**, the representations of the discrete series are given by a natural number n. Half of them are realized in the space H_n of all functions $\varphi(z)$, $z = x + iy$, that are analytic in the upper half-plane and for which

$$\int\limits_{\operatorname{Im} z > 0} |\varphi(z)|^2 y^{n-1}\, dx\, dy < \infty.$$

The other half of the representations of the discrete series are realized in the space of functions that are analytic in the lower half-plane. The representation $T_n(g)$ is given by the following formula:

$$T_n(g)\,\varphi(z) = \varphi\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right)(g_{12}z + g_{22})^{-n-1}. \tag{1}$$

For the sake of precision we shall now discuss only the first half of the representations of the discrete series.

Instead of the space H_n we consider the space Ω_n of *all* functions that are analytic in the upper half-plane and endow Ω_n with

the natural topology. So we obtain a complete space in which the representation of G defined by (1) also acts. It can be shown that this representation is (topologically) irreducible.

An automorphic form for the representation $T_n(g)$ of the discrete series is defined as a function $\varphi(z)$ that is analytic in the upper half-plane and invariant under the operators $T_n(\gamma)$, that is, for which

$$\varphi\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}\right)(\gamma_{12}z + \gamma_{22})^{-n-1} = \varphi(z) \tag{2}$$

for every $\gamma \in \Gamma$. (This function need not lie in the Hilbert space H_n.)

Automorphic forms for the second half of the discrete series are defined similarly.

Now we pass on to the representations of the principal and supplementary series. When we speak here of representations of the principal series, we have in mind only the representations $T_s(g)$ that are realized in the spaces of *even* functions on the affine plane (that is, those representations for which $T_s(-g) = T_s(g)$).

From **§ 3.4** we know that a representation $T_s(g)$ of the principal or supplementary series may be realized in a certain subspace of the functions $\varphi(z)$ defined on the upper half-plane $\operatorname{Im} z > 0$ and satisfying the equation

$$\Delta\varphi \equiv -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi = \frac{1 - s^2}{4}\varphi. \tag{3}$$

(We recall that the Laplace operator Δ commutes with the fractional-linear transformations of the half-plane.)

The representation operator of $T_s(g)$ is defined by the following formula:

$$T_s(g)\varphi(z) = \varphi\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right). \tag{4}$$

Note that equivalent representations correspond to the numbers s and $-s$; thus, the representation $T_s(g)$ is uniquely determined by the eigenvalue $\dfrac{1 - s^2}{4}$ of the Laplace operator Δ.

Now we introduce the space Ω_s of all functions $\varphi(z)$ defined in the upper half-plane and satisfying equation (3).† The operator $T_s(g)$ defined by (4) gives the representation of G in this space Ω_s. An automorphic form for the representation $T_s(g)$ of the principal or supplementary series is defined as a function $\varphi(z) \in \Omega_s$ that is invariant under the operators $T_s(\gamma)$, $\gamma \in \Gamma$, that is, for which

† Since (3) is an elliptic equation, these functions are infinitely differentiable.

$T_s(\gamma)\varphi(z) = \varphi(z)$. A fractional automorphic form corresponding to the representation $T_s(g)$ is a function $\varphi(z)$ defined in the upper half-plane and satisfying the following condition:

$$\Delta\varphi \equiv -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi = \frac{1-s^2}{4}\varphi, \qquad \varphi\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}\right) = \varphi(z)$$

for every $\gamma \in \Gamma$.

2. Statement of the Duality Theorem. Let Γ be a discrete subgroup of the group G of real unimodular matrices of order 2 for which $X = \Gamma \setminus G$ is a compact space.

We consider the representation of G generated by X. We recall that it is constructed in the space of functions $f(x)$, $x \in X$, for which

$$\int |f(x)|^2\, dx < \infty,$$

where dx is the invariant measure on X. The representation operator $T(g)$ is given by the following formula:

$$T(g)f(x) = f(xg).$$

The representation $T(g)$ splits into representations of the principal continuous, the supplementary, and the discrete series.

For simplicity we assume that Γ contains the matrix $g_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the decomposition of $T(g)$ does not contain representations H_s^- of the principal continuous series realized in the spaces of homogeneous odd functions on the affine plane.

For under our assumptions $T(g_0)$ is the unit operator: $T(g_0)f(x) = f(x)$, whereas for the representations H_s^- we have $T(g_0)f = -f$. For the same reason the decomposition of $T(g)$ does not contain representations of the discrete series with an even index n.

Since X is a compact space, $T(g)$ splits (see **§ 2**) into the sum of a countable number of irreducible representations, and each representation occurs in the decomposition with finite multiplicity. We want to know what irreducible representations actually occur in this decomposition and with what multiplicity. The answer is given by the following theorem.

Duality Theorem. *Each of the irreducible representations of G occurs in $T(g)$ with finite multiplicity, equal to the dimension of the space of the corresponding automorphic forms.*

In other words, if $T_s(g)$ is a representation of the fundamental or the supplementary series, then the multiplicity with which $T_s(g)$ occurs in $T(g)$ is equal to the number of linearly independent

functions $\varphi(z)$ defined on the half-plane $\operatorname{Im} z > 0$ and satisfying the condition

$$\Delta\varphi(z) \equiv -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi = \frac{1-s^2}{4}\varphi, \qquad \varphi\left(\frac{\gamma_{11}z+\gamma_{21}}{\gamma_{12}z+\gamma_{22}}\right) = \varphi(z)$$

for every $\gamma \in \Gamma$. But if $T_n(g)$ is a representation of the discrete series (acting, for example, in the space of functions analytic in the upper half-plane), then the multiplicity with which $T_n(g)$ occurs in $T(g)$ is equal to the number of functions $\varphi(z)$ that are analytic in the upper half-plane and satisfy the condition

$$\varphi\left(\frac{\gamma_{11}z+\gamma_{21}}{\gamma_{12}z+\gamma_{22}}\right)(\gamma_{12}z+\gamma_{22})^{-n-1} = \varphi(z)$$

for every $\gamma \in \Gamma$. The proof of the duality theorem will be given in the next subsection.

3. The Laplace Operator. An important role in the proof of the duality theorem is played by the Laplace operator Δ. Let $T(g)$ be any unitary representation of G. We consider the following one-parameter subgroups of G:

$$\left.\begin{aligned} &g_1(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \qquad g_2(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \\ &g_3(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \end{aligned}\right\} \tag{1}$$

To these subgroups there correspond one-parameter groups of unitary operators in the representation space H of $T(g)$:

$$T_k(t) = T(g_k(t)), \qquad k = 1, 2, 3. \tag{2}$$

By the well-known Stone theorem, every one-parameter group $T_k(t)$ of unitary operators is of the form:

$$T_k(t) = e^{itU_k}, \tag{3}$$

where U_k is a self-adjoint operator in H.

We define the operator Δ by the following formula:

$$\Delta = -\tfrac{1}{4}(U_1^2 - U_2^2 - U_3^2) \tag{4}$$

and call Δ the Laplace operator in H.

Observe that the operators U_k and Δ are defined not on the whole space H. However, they are defined on a certain everywhere

dense linear manifold in H. For let us call a vector $f \in H$ infinitely differentiable if $T(g)f$ is an infinitely differentiable vector function of g. The set of infinitely differentiable vectors in H is called the *Gårding space*. It can be shown that the Gårding space is everywhere dense in H and that iU_1, iU_2, iU_3 and Δ are symmetric operators in the Gårding space (see, for example, Nelson [48]).

Let us compute the operator Δ on each irreducible invariant subspace of H.

We show that on the Gårding space of each irreducible space H the operator Δ is a multiple of the unit operator E.† Specifically, in the space of a representation $T_s(g)$ of the principal or supplementary series $\Delta = \dfrac{1 - s^2}{4} E$; in the space of a representation of the discrete series of index n, $\Delta = \dfrac{1 - n^2}{4} E$.

First we examine a representation $T_s^+(g)$ of the principal or supplementary series. From **§ 3.3** and **§ 3.4** we know that this representation can be realized in a certain space of infinitely differentiable functions $\varphi(z)$ defined on the half-plane Im $z > 0$ and satisfying the equation

$$(z - \bar{z})^2 \frac{\partial^2 \varphi}{\partial z\, \partial \bar{z}} = \frac{1 - s^2}{4} \varphi. \tag{5}$$

Here the representation operator of $T_s^+(g)$ has the form:

$$T_s^+(g)\, \varphi(z) = \varphi\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right). \tag{6}$$

When we substitute in (6) for g the matrix $g_k(t)$ and differentiate these expressions with respect to t, we obtain the following formulae for the operators U_1, U_2, U_3:

$$iU_1 = -(1 + z^2) \frac{\partial}{\partial z} - (1 + \bar{z}^2) \frac{\partial}{\partial \bar{z}},$$

$$iU_2 = (1 - z^2) \frac{\partial}{\partial z} + (1 - \bar{z}^2) \frac{\partial}{\partial \bar{z}},$$

$$iU_3 = 2z \frac{\partial}{\partial z} + 2\bar{z} \frac{\partial}{\partial \bar{z}}.$$

From this we find immediately that

$$\Delta \equiv -\tfrac{1}{4}(U_1^2 - U_2^2 - U_3^2) = (z - \bar{z})^2 \frac{\partial^2}{\partial z\, \partial \bar{z}}.$$

Consequently, by (5) we have $\Delta = \dfrac{1 - s^2}{4} E$.

† More accurately, $\Delta = \lambda E$ on the Gårding subspace.

Similar arguments hold for the representations $T_s^-(g)$ of the second fundamental series.

Now we consider a representation of the discrete series $T_n(g)$ contained in $T(g)$. This representation is realized in the space of functions $\varphi(z)$ that are analytic either in the upper half-plane or in the lower half-plane. For the sake of precision, we shall now consider the first case.

The representation operator of $T_n(g)$ takes the following form:

$$T_n(g)\,\varphi(z) = \varphi\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right)(g_{12}z + g_{22})^{-n-1}. \tag{7}$$

Substituting in (7) for g the matrices $g_k(t)$ and differentiating with respect to t we obtain the following formulae for the operators U_1, U_2, U_3:

$$iU_1 = -(1 + z^2)\frac{\partial}{\partial z} - (n + 1)z,$$

$$iU_2 = (1 - z^2)\frac{\partial}{\partial z} - (n + 1)z,$$

$$iU_3 = 2z\frac{\partial}{\partial z} + (n + 1).$$

From this we find immediately that

$$\Delta \equiv -\tfrac{1}{4}(U_1^2 - U_2^2 - U_3^2) = \frac{1 - n^2}{4}E.$$

4. Proof of the Duality Theorem for Representations of the Continuous Series. Let $T_s(g)$ be a representation of a principal or supplementary series. We have to show that the multiplicity with which this representation occurs in the representation $T(g)$ generated by the space $X = \Gamma \setminus G$ is equal to the largest number of corresponding linearly independent automorphic forms, that is, to the largest number of linearly independent functions $\varphi(z)$ defined on the half-plane $\operatorname{Im} z > 0$ and satisfying the following condition:

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi = \frac{1 - s^2}{4}\varphi,$$

$$\varphi\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}\right) = \varphi(z)$$

for every $\gamma \in \Gamma$.

First we show that the multiplicity with which a representation $T_s(g)$ of the fundamental or supplementary series is contained in $T(g)$ is equal to the largest number of linearly independent functions

on $X = \Gamma \setminus G$ that satisfy the following condition:

$$\Delta f = \frac{1 - s^2}{4} f, \qquad T(g_1(t))f = f. \tag{1}$$

Here Δ is the Laplace operator constructed in **§ 4.3**, and

$$g_1(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

so that $g_1(t)$ is an orthogonal matrix.

For let H be the space on which $T(g)$ acts, and H_s^+ its irreducible subspace on which the representation equivalent to $T_s(g)$ acts. We know that H_s^+ contains one and, to within a constant factor, only one vector f for which $T(g_1(t))f = f$ (see **§ 3.4.**); on this vector the operator Δ is defined, and $\Delta f = \dfrac{1 - s^2}{4} f$.

On the other hand, suppose that the vector $f \in H$ satisfies the relations (1). Splitting H into a direct sum of irreducible subspaces relative to $T(g)$, we have $f = \sum f_k$, where† $f_k \in H_{s_k}^+$. Then the vectors f_k also satisfy the relation $T(g_1(g))f_k = f_k$, the operator Δ is defined on them and $\Delta f_k = \dfrac{1 - s_k^2}{4}$. Since $(\Delta f, f_k) = (f, \Delta f_k)$, we have $\dfrac{1 - s_k^2}{4} = \dfrac{1 - s^2}{4}$. Our assertion follows immediately from this.

Now let us find the largest number of linearly independent functions f satisfying (1). For this purpose we introduce a convenient parametrization of G and extend the functions from X to the group G.

Let $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$. As defining parameters for the matrix g we take the complex number $z = -\dfrac{g_{22} + ig_{21}}{g_{12} + ig_{11}}$ and the real number $\theta = \arg(g_{12} - ig_{11})$.

We note that $\operatorname{Im} z = \dfrac{g_{11}g_{22} - g_{12}g_{21}}{|g_{12} + ig_{11}|^2} = |g_{12} + ig_{11}|^{-2}$, so that $\operatorname{Im} z > 0$.

It is easy to verify that the parameters z and θ transform according to the following formulae:

1. If $g = (z, \theta)$, $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $a^{-1}g = (z_1, \theta_1)$, where

$$z_1 = \frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}, \qquad \theta_1 = \theta - \arg(a_{12}z + a_{22}).$$

† The components of f in the irreducible subspaces in which the representations of the discrete series act are equal to zero, because in these subspaces there are no vectors invariant under $T(g_1(t))$.

2. If $u = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$, then $gu = (z, \theta + \varphi)$.

The functions on $X = \Gamma \setminus G$ can be regarded as functions $f(g)$ on the whole group G that satisfy the condition $f(\gamma^{-1}g) = f(g)$ for all $\gamma \in \Gamma$; in other words, as functions $f(z, \theta)$ which satisfy the functional equation

$$f\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}, \theta - \arg(\gamma_{12}z + \gamma_{22})\right) = f(z, \theta)$$

for all $\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$ in Γ. Thus, the conditions (1) may be written in the following form:

$$f(z, \theta + \varphi) = f(z, \theta); \tag{2}$$

$$\Delta f = \frac{1 - s^2}{4} f; \tag{3}$$

$$f\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}, \theta - \arg(\gamma_{12}z + \gamma_{22})\right) = f(z, \theta). \tag{4}$$

From (2) it follows that f does not depend on θ, that is, f is a function of the complex variable z only, $\operatorname{Im} z > 0$.

Next, by a simple calculation we find that the operator Δ assumes in the coordinates x, y, θ ($z = x + iy$) the following form:†

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + h\,\frac{\partial^2}{\partial x\,\partial \theta}.$$

† We indicate the computation of Δ. The operator of the representation $T(a)$ is given by the formula $T(a)f(a) = f(ga)$, $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, or in the parameters z and θ

$$T(a)f(z, \theta) = f(z_1, \theta_1), \tag{5}$$

where

$$z_1 = -\frac{g_{22}(a_{22} + ia_{21}) + g_{21}(a_{12} + ia_{11})}{g_{12}(a_{22} + ia_{21}) + g_{11}(a_{12} + ia_{11})},$$

$$\theta_1 = \arg(g_{12}(a_{22} - ia_{21}) + g_{11}(a_{12} - ia_{11})).$$

When we substitute in (5) for a the matrices $g_k(t)$, $k = 1, 2, 3$ (see p. 48) and differentiate with respect to t, we obtain the following formulae for the operators U_1, U_2, U_3:

$$iU_1 = \frac{\partial}{\partial\theta}, \; iU_2 = 2y\left(\cos 2\theta\,\frac{\partial}{\partial x} + \sin 2\theta\,\frac{\partial}{\partial y}\right) - \cos 2\theta\,\frac{\partial}{\partial\theta},$$

$$iU_3 = 2y\left(-\sin 2\theta\,\frac{\partial}{\partial x} + \cos 2\theta\,\frac{\partial}{\partial y}\right) + \sin 2\theta\,\frac{\partial}{\partial\theta}.$$

Hence we find that

$$\Delta = -\tfrac{1}{4}(U_1^2 - U_2^2 - U_3^2) = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\,\frac{\partial^2}{\partial x\,\partial\theta}.$$

Consequently, since the functions f do not depend on θ, condition (3) takes the form

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f = \frac{1-s^2}{4}f.$$

So we have established that the largest number of linearly independent solutions of (1) is equal to the largest number of linearly independent solutions of the equations

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f(z) = \frac{1-s^2}{4}f(z),$$

$$f\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}\right) = f(z),$$

that is, to the largest number linearly independent of automorphic forms of the corresponding irreducible representation. This proves the duality theorem for representations of the principal or supplementary series.

5. Proof of the Duality Theorem for Representations of the Discrete Series. Let $T_n(g)$ be a representation of the discrete series and let $T_n(g)$ be contained in $T(g)$. For the sake of definiteness we assume that it is realized in the space of functions $\varphi(z)$ analytic in the upper half-plane. The representation operator of $T_n(g)$ has the following form:

$$T_n(g)\,\varphi(z) = \varphi\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right)(g_{12}z + g_{22})^{-n-1}. \qquad (1)$$

We have to show that the multiplicity with which this representation occurs in $T(g)$ is equal to the largest number of corresponding linearly independent automorphic forms, that is, to the largest number of linearly independent functions $\varphi(z)$ that are analytic in the half-plane $\operatorname{Im} z > 0$ and satisfy the condition

$$\varphi\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}\right)(\gamma_{12}z + \gamma_{22})^{-n-1} = \varphi(z)$$

for every $\gamma \in \Gamma$.

To begin with we show that the multiplicity with which a representation $T_n(g)$ of the discrete series is contained in $T(g)$ is equal to the largest number of linearly independent functions on $X = \Gamma \setminus G$ that satisfy the following conditions:

$$\Delta f = \frac{1-n^2}{4}f, \qquad T(g_1(t))f = e^{-i(n+1)t}f. \qquad (2)$$

Here Δ is the Laplace operator constructed in **§ 4.3** and

$$g_1(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

For the proof we remark that on the one hand, every irreducible subspace H_n in which $T_n(g)$ acts has one and, to within a constant factor, only one vector f for which

$$T_n(g_1(t))f = e^{-i(n+1)t}f. \qquad (3)$$

For in the model (1) of $T_n(g)$ the condition (3) can be written in the following form:

$$\varphi\left(\frac{z\cos t - \sin t}{z\sin t + \cos t}\right)(z\sin t + \cos t)^{-n-1} = e^{-i(n+1)t}\varphi(z). \qquad (4)$$

It is not hard to check that (4) is satisfied by the function

$$\varphi(z) = (z+i)^{-n-1}$$

and that this is the only solution of (4) in the space of functions analytic in the upper half-plane.

We call this vector f a *vector of dominant weight* in H_n. On this vector the operator Δ is defined, and

$$\Delta f = \frac{1-n^2}{4}f.$$

On the other hand, we can show that every vector $f \in H$ satisfying (2) is a linear combination of vectors of dominant weight from irreducible subspaces of H equivalent to H_n. For when we split H into a direct sum of irreducible subspaces relative to $T(g)$, we have $f = \Sigma f_k$ where $f_k \in H_{s_k}$. Then the vectors f_k also satisfy the relation $T(g_1(t))f_k = e^{-i(n+1)t}f_k$, the operator Δ is defined on them and $\Delta f_k = \frac{1-s_k^2}{4}f_k$. Since $(f, \Delta f_k) = (\Delta f, f_k)$, we have

$$\frac{1-s_k^2}{4} = \frac{1-n^2}{4};$$

consequently the f_k are vectors of dominant weight in subspaces equivalent to H_n.† Hence our assertion follows immediately.

Now let us find the largest number of linearly independent functions f satisfying (2). Repeating the arguments on p. 51, instead of the relations (2), (3), and (4) on p. 52 we obtain the following relations for $f(z, \theta)$:

$$f(z, \theta + \varphi) = f(z, \theta)e^{-i(n+1)\varphi}, \qquad (5)$$

$$\Delta f \equiv \left[-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x\,\partial\theta}\right]f = \frac{1-n^2}{4}f, \qquad (6)$$

$$f\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}, \theta - \arg(\gamma_{12}z + \gamma_{22})\right) = f(z, \theta). \qquad (7)$$

† This statement must be made more accurate, because there are two representations of the discrete series with one and the same index n: a representation in the space of functions analytic in the upper half-plane, and one in the space of functions analytic in the lower half-plane. However, it is easy to check that a vector satisfying (3) exists in only the first of these spaces.

Let us find all such functions. First of all, from (5) it follows that

$$f(z, \theta) = e^{-i(n+1)\theta} f_1(z).$$

We represent a solution of the equations (6) and (7) in the form

$$f(z, \theta) = e^{-i(n+1)\theta} \overline{y^{\frac{n+1}{2}} F(z)}. \tag{8}$$

Substituting this expression in (6) and (7) we obtain the following relations for the functions $F(z)$:

$$\left[-y\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i(n+1)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]F(z) = 0. \tag{9}$$

$$F\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}\right) = F(z)(\gamma_{12}z + \gamma_{22})^{n+1} \tag{10}$$

for every $\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$ in Γ.

The relation (10) means that the function $F(z)$ satisfies the functional equation for an automorphic form.

Let us show that $F(z)$ is an analytic function.

Suppose that $f \in H$ satisfies the conditions (5) through (7). As we have shown earlier, such a function f may be represented in the form $f = \sum f_k$, where the f_k are vectors of dominant weight in irreducible subspaces of H equivalent to H_n. (The number of terms of this sum is finite, since each irreducible subspace occurs in H with finite multiplicity.)

We introduce the operator

$$A_+ = iU_2 + U_3,$$

where U_2, U_3 are the self-adjoint operators defined on p. 48. The operator A_+ is defined on each of the vectors f_k of dominant weight, and hence also on $f = \sum f_k$. We show that $A_+ f_k = 0$ for every f_k, so that

$$A_+ f = 0.$$

We realize the irreducible subspace H_n of H as the space of functions analytic in the upper half-plane. In this realization the vector f_k of dominant weight has the form:

$$f_k = c(z+i)^{-n-1}. \tag{11}$$

But the operator A_+ is given in H_n by the following formula:

$$A_+ = -(z+i)^2 \frac{\partial}{\partial z} - (n+1)(z+i). \tag{12}$$

(This formula follows from the expressions for the operators U_2 and

U_3 in H_n that were given on p. 50.) From (11) and (12) we obtain immediately that $A_+f_k = 0$.

We can write down an explicit expression for A_+ in the space of functions $f(z, \theta)$. On the basis of the formulae for U_2 and U_3 on p. 52 we find

$$A_+ = e^{2i\theta}\left[2y\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) - \frac{\partial}{\partial \theta}\right].$$

When we substitute in the equation $A_+f = 0$ for f its expression (8), we obtain

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)F = 0.$$

So we have shown that $F(z)$ is an analytic function. It is clear that, conversely, if $F(z)$ is an analytic function, then it satisfies (9). But then the function f defined by (8) satisfies (6).

We have now proved that the conditions (5)–(7) on the function f are equivalent to the following conditions on the function $F(z)$, which is connected with f by (8):

1. $F(z)$ satisfies the functional equation

$$F\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}\right)(\gamma_{12}z + \gamma_{22})^{-n-1} = F(z);$$

2. $F(z)$ is analytic in the upper half-plane.

In other words, $F(z)$ is an automorphic form for the representation $T_n(g)$ of the discrete series. So we have shown that the multiplicity with which $T_n(g)$ occurs in $T(g)$ is equal to the number of linearly independent forms corresponding to this representation $T_n(g)$. The proof of the duality theorem is now complete.

REMARK 1. The duality theorem has been proved here under the assumption that Γ contains the matrix $g_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Under this assumption the decomposition of the representation of G generated by the space $X = \Gamma \setminus G$ does not contain the representations H_s^- of the principal continuous series (that is, the representations realized in the space of odd functions on the affine plane), nor the representations of the discrete series with even index n.

It can be shown that the duality theorem remains valid even when Γ does not contain the matrix $g_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that in this case the decompositions of the representation of G generated by the space $X = \Gamma \setminus G$ also contains the representations H_s^- and the representations of the discrete series with an even index n.

Let us give a definition of an automorphic form for the representation H_s^-.

We consider the space of continuous functions $\varphi(z)$ on the half-plane $\operatorname{Im} z > 0$. In this space we give a representation $T(g)$ of G by the following formula:

$$T(g)\varphi(z) = \varphi\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right)(g_{12}z + g_{22})^{-1}. \tag{15}$$

A differential operator of the second order Δ in the space of functions $\varphi(z)$ that commutes with all the operators $T(g)$ has the following form:

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iy\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{16}$$

It can be shown that the representation H_s^- may be realized in a certain subspace of functions $\varphi(z)$ satisfying the equation

$$\Delta\varphi(z) = \frac{1 - s^2}{4}\varphi(z). \tag{17}$$

Here the representation operator is given by (15). An automorphic form corresponding to the representation H_s^- is defined as any function $\varphi(z)$ satisfying (17) and the relation

$$\varphi\left(\frac{\gamma_{11}z + \gamma_{21}}{\gamma_{12}z + \gamma_{22}}\right)(\gamma_{12}z + \gamma_{22})^{-1} = \varphi(z)$$

for every $\gamma \in \Gamma$.

The proof of the duality theorem for the representations H_s^- is an almost verbatim repetition of that for the representations H_s^+. We leave this as an exercise for the reader.

REMARK 2. Without proof we give a statement of the duality theorem for an arbitrary representation $T(g)$, induced by a finite-dimensional representation of Γ.

Let Γ be a discrete subgroup of G such that the space $X = \Gamma \setminus G$ is compact, $\chi(\gamma)$ a finite-dimensional irreducible unitary representation of Γ, and $T(g)$ the representation of G induced by it.

Then the multiplicity with which a given irreducible representation $T_k(g)$ of G occurs in $T(g)$ is equal to the multiplicity with which the space of the irreducible representation $\chi(\gamma)$ of Γ is contained in the space Ω_k of the representation T_k.

The definition of the spaces Ω_k for various irreducible representations of G was given above.

6. The General Duality Theorem. In this subsection we give a proof of the duality theorem for an arbitrary semisimple Lie group G. Every Hilbert space H in which an irreducible unitary representation of G acts can be imbedded in a certain linear topological space Ω. The duality theorem will then be established in the

following form: *if the factor space* $X = \Gamma \setminus G$ *is compact, then the multiplicity with which an irreducible representation H occurs in the space generated by X is equal to the largest number of linearly independent vectors in* Ω *that are invariant under* Γ.

In the construction of Ω we need the following property (A) of representations of semisimple Lie groups, which is stated below without proof.

Let G be a semisimple Lie group, U its maximal compact subgroup, H the Hilbert space in which an irreducible unitary representation $T(g)$ of G acts.

In H we consider the representation of U. Since U is compact, H can be decomposed into the direct sum of subspaces H_m such that in each the representation of U that acts in it is a multiple of an irreducible one. (It is assumed that the irreducible representations of U in the spaces H_m are inequivalent.)

Each of the spaces H_m *is finite-dimensional. In other words, every irreducible representation of U occurs in H with finite multiplicity.*†

The elements ξ from the subspaces H_m and their finite linear combinations are called *U-polynomials*. Clearly, $\xi \in H$ is a U-polynomial if and only if the space spanned by the vectors $T(u)\xi$, $u \in U$, is finite-dimensional.

We now turn to the construction of the space Ω.

In H we fix a certain U-polynomial ξ_0. With every vector $\eta \in H$ we associate a continuous function on G:

$$\varphi_\eta(g) = (T(g)\eta, \xi_0), \tag{1}$$

where $(.\,,.)$ denotes the scalar product in H.

It is easy to see that the correspondence

$$\eta \to \varphi_\eta(g) \tag{2}$$

is linear and that $\varphi_\eta(g) = 0$ identically on G only when $\eta = 0$. So the correspondence is an isomorphism.

There is a natural topology in the set of functions $\varphi_\eta(g)$: a sequence of functions is called convergent if it converges uniformly on every compact subset of G. We extend this topology to H and complete H in this topology. The space so obtained is denoted by Ω.

In the construction of Ω the choice of the U-polynomial ξ_0 was arbitrary. We shall now show that in fact Ω does not depend on the choice of ξ_0. For this purpose it is enough to prove the following assertion: *If for one* $\xi_0 \in H$ *the sequence of functions*

$$\varphi_{\eta_n,\xi_0}(g) = (T(g)\eta_n, \xi_0) \tag{3}$$

converges uniformly on every compact set in G, then the same is true for every U-polynomial ξ.

† It is precisely this property, and not the semisimplicity of G, that is used in the proof of the duality theorem.

Proof. If the sequence $\varphi_{\eta_n, \xi_0}(g)$ converges on every compact set in G, then this also holds for every vector ξ of the form

$$\xi = \sum_{k=1}^{n} c_k T(g_k) \xi_0. \tag{4}$$

These vectors ξ form an everywhere dense subset H_0 of H.

Let $\xi_m \neq 0$ be an arbitrary U-polynomial belong to H_m. From the fact that H_0 is an everywhere dense linear subset of H and that H_m is a finite-dimensional subspace of H it follows that the projection of H_0 onto H_m coincides which the whole space H_m. In other words, there exists a vector ξ of the form (4) such that

$$P_m \xi = \xi_m. \tag{5}$$

where P_m is the projection operator sending H_0 onto H_m.

The projection operator P_m is given by the following formula:

$$P_m \xi = \int_U \overline{\sigma_m(u)}\, \xi \, du. \tag{6}$$

where $\sigma_m(u)$ is the character of the irreducible representation of U corresponding to H_m.

So we obtain from (5) and (6)

$$\varphi_{\eta_n, \xi_m}(g) \equiv (T(g)\eta_n, P_m \xi) = \int_U \sigma_m(u) (T(u^{-1}g)\eta_n, \xi) \, du,$$

that is,

$$\varphi_{\eta_n, \xi_m}(g) = \int_U \sigma_m(u) \, \varphi_{\eta_n, \xi}(u^{-1}g) \, du. \tag{7}$$

From the fact that the sequence of functions $\varphi_{\eta_n, \xi}(g)$ converges uniformly on every compact set in G we deduce, by (7), that the sequence of functions $\varphi_{\eta_n, \xi_m}(g)$ has the same property. This proves the assertion.

In this way we place every Hilbert space H, in which an irreducible unitary representation of G acts, in a complete linear topological space Ω.

We now introduce the concept of a matrix element in Ω.

To begin with, let η, $\xi \in H$. Then we set

$$f_{\eta, \xi}(g) = (T(g)\eta, \xi), \tag{8}$$

where $(.\,,.)$ denotes the scalar product in H.

Now let η and ξ be vectors from Ω of which one, for example ξ, belongs to H. We assume that there exists a sequence $\{\eta_n\}$ of vectors from H that converges to η in the topology of Ω and such that the sequence of functions

$$f_{\eta_n, \xi}(g) = (T(g)\eta_n, \xi)$$

converges uniformly on every compact set of G. Then we set

$$f_{\eta,\xi}(g) = \lim_{n\to\infty} f_{\eta_n,\xi}(g). \tag{9}$$

Similarly we define $f_{\eta,\xi}(g)$ for $\eta \in H$. The function $f_{\eta,\xi}(g)$ is called a *matrix element* in Ω.

We now establish the main properties of these functions. From the definition it follows immediately that $f_{\eta,\xi}(g)$ is continuous function of g. Furthermore, it is not difficult to verify the following properties of the matrix elements $f_{\eta,\xi}(g)$:

1. For every η there is at least one ξ for which $f_{\eta,\xi}(g)$ is defined. (In fact, $f_{\eta,\xi}(g)$ is defined if one of the vectors ξ or η is a U-polynomial—this follows from the definition of Ω.)

2. If $f_{\eta,\xi}(g)$ is defined, then $f_{\xi,\eta}(g)$ is also defined, and $f_{\eta,\xi}(g) = \overline{f_{\eta,\xi}(g^{-1})}$.

3. If $f_{\eta_1,\xi}(g)$ and $f_{\eta_2,\xi}(g)$ are defined, then $f_{\alpha_1\eta_1+\alpha_2\eta_2,\xi}(g)$ is also defined, and

$$f_{\alpha_1\eta_1+\alpha_2\eta_2,\xi}(g) = \alpha_1 f_{\eta_1,\xi}(g) + \alpha_2 f_{\eta_2,\xi}(g).$$

4. If $f_{\xi,\xi}(g)$ exists, then $f_{\xi,\xi}(e) > 0$, where e denotes the unit element of G.

5. If $f_{\eta,\xi}(g)$ exists, then $f_{T(g_1)\eta,T(g_2)\xi}(g)$ exists for arbitrary g_1, $g_2 \in G$, and

$$f_{T(g_1)\eta,T(g_2)\xi}(g) = f_{\eta,\xi}(g_2^{-1}gg_1).$$

6. Let η_n be a sequence that converges to η in the topology of Ω. If the $f_{\eta_n,\xi}(g)$ are defined and are uniformly convergent on every compact set in G, then $f_{\eta,\xi}(g)$ is also defined and

$$f_{\eta,\xi}(g) = \lim_{n\to\infty} f_{\eta_n,\xi}(g).$$

Finally, the space Ω has the following completeness property. *If for some $\xi \in H$ the sequence of functions $f_{\eta_n,\xi}(g)$ is defined and is uniformly convergent on every compact set in G, then the sequence η_n converges in Ω.*

For by virtue of the proposition on p. 58 if the sequence $f_{\eta_n,\xi}(g)$ is uniformly convergent on every compact subset of G, then the same is true for the sequence $f_{\eta_n,\xi_0}(g)$, where ξ_0 is an arbitrary U-polynomial. But then, by the definition of the topology in Ω, the sequence η_n is convergent.

Now we pass on to the statement and proof of the duality theorem.

Let G be a semisimple Lie group, and Γ a discrete subgroup of it for which the factor space $X = \Gamma \setminus G$ is compact, H the space of an irreducible unitary representation of G, and Ω the linear topological space containing H, as constructed above.

DUALITY THEOREM. *The multiplicity with which H occurs in the representation of G generated by X is equal to the largest number of linearly independent vectors in Ω that are invariant under Γ.*

Proof. We assume that the representation generated by X is realized in the space $\mathscr{H}$ of functions on G that satisfy the following conditions:

$$f(\gamma g) = f(g) \quad \text{for every} \quad \gamma \in \Gamma;$$

$$\|f\|^2 \equiv \int_F |f(g)|^2 \, dg < \infty,$$

where F is a fundamental domain in G relative to Γ. The representation operator $T(g)$ is given by the formula

$$T(g_0)f(g) = f(gg_0).$$

Suppose that the space H in which the irreducible unitary representation of G acts occurs in $\mathscr{H}$. This means that we have a correspondence: with every $\xi \in H$ is associated a function $\varphi_\xi(g) \in \mathscr{H}$, and

1. $\varphi_{\alpha_1\xi_1+\alpha_2\xi_2}(g) = \alpha_1\varphi_{\xi_1}(g) + \alpha_2\varphi_{\xi_2}(g)$,
2. $\varphi_{T(g_0)\xi}(g) = \varphi_\xi(gg_0)$ for arbitrary g_0, $g \in G$,
3. $(\xi_1, \xi_2) = \int_F \varphi_{\xi_1}(g)\overline{\varphi_{\xi_2}(g)} \, dg.$

Observe that the set of functions $\varphi_\xi(g)$ contains an everywhere dense subset of continuous functions. For this set contains together with $\varphi_\xi(g)$ also the function $\varphi_{A\xi}(g)$, where $A = \int_U T(g) \, dg$, U being an arbitrary neighborhood of the unit element of G. Clearly the functions $\varphi_{A\xi}(g)$ form an everywhere dense subset. On the other hand, from the formula

$$\varphi_{A\xi}(g) = \int_U \varphi_\xi(gg_1) \, dg_1$$

it follows easily that these functions are continuous.

Now we associate with the space H a vector $\eta \in \Omega$ that is invariant under the operators $T(\gamma)$, $\gamma \in \Gamma$. For this purpose we give a basis of neighborhoods of the unit element of G:

$$U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots.$$

We consider the averages of the functions $\varphi_\xi(g)$ on each of these neighborhoods:

$$\psi_{\xi,n}(g) = \frac{1}{\operatorname{mes} U_n} \int_{U_n} \varphi_\xi(g_1 g) \, dg_1.$$

It is easy to verify that $\psi_{\xi,n}(e)$ is a continuous linear functional on H. Consequently, for every n there exists a vector $\eta_n \in H$ such that

$$\psi_{\xi,n}(e) = (\xi, \eta_n).$$

Hence it follows that

$$\psi_{\xi,n}(g) = \psi_{T(g)\xi,n}(e) = (T(g)\xi, \eta_n),$$

that is,

$$\psi_{\xi,n}(g) = f_{\xi,\eta_n}(g).$$

If $\varphi_\xi(g)$ is continuous, then the sequence of functions $\psi_{\xi,n}(g) = f_{\xi,\eta_n}(g)$ is uniformly convergent to $\varphi_\xi(g)$ on every compact set in G. Consequently, by the completeness property, the sequence of the corresponding vectors $\eta_n \in H$ converges in the topology of Ω to a vector η and

$$f_{\xi,\eta}(g) = \varphi_\xi(g).$$

This vector η is invariant under the operators $T(\gamma)$, $\gamma \in \Gamma$. For we have

$$f_{\xi,T(\gamma)\eta}(g) = f_{\xi,\eta}(\gamma^{-1}g) = \varphi_\xi(\gamma^{-1}g) = \varphi_\xi(g) = f_{\xi,\eta}(g).$$

Consequently, $T(\gamma)\eta = \eta$ for all $\gamma \in \Gamma$.

So we have associated with the irreducible subspace H contained in $\mathscr{H}$ a vector $\eta \in \Omega$ that is invariant under $T(\gamma)$, $\gamma \in \Gamma$. Now we prove the converse: to every vector $\eta \in \Omega$ that is invariant under all $T(\gamma)$, $\gamma \in \Gamma$, there corresponds a realization of the space H in $\mathscr{H}$.

For let η_0 be an invariant vector in Ω. If ξ is an U-polynomial, then $f_{\xi,\eta_0}(g)$ is defined. We set

$$\varphi_\xi(g) = f_{\xi,\eta_0}(g).$$

It is easy to see that

$$\varphi_\xi(\gamma^{-1}g) = \varphi_\xi(g)$$

for every $\gamma \in \Gamma$. For we have

$$\varphi_\xi(\gamma^{-1}g) = f_{\xi,\eta_0}(\gamma^{-1}g) = f_{\xi,T(\gamma)\eta_0}(g) = f_{\xi,\eta_0}(g) = \varphi_\xi(g).$$

We introduce a scalar product in the set of functions $\varphi_\xi(g)$ by the formula

$$(\varphi_{\xi_1}(g), \varphi_{\xi_2}(g)) = \int_F \varphi_{\xi_1}(g)\overline{\varphi_{\xi_2}(g)}\, dg.$$

By completing the set of functions $\varphi_\xi(g)$ with respect to this scalar product we obtain a realization of H in the form of a subspace of $\mathscr{H}$.

In this way we have associated with the vector $\eta \in \Omega$ invariant under all $T(\gamma)$, $\gamma \in \Gamma$, a realization of H in $\mathscr{H}$. It is easy to verify that linearly independent vectors η correspond to linearly independent spaces. This shows that the multiplicity with which η occurs

in $\mathscr{H}$ is equal to the largest number of linearly independent vectors in Ω that are invariant under Γ. The proof of the duality theorem is now complete.

In the discussion of the group of unimodular matrices of order 2, the space Ω is defined in a different way, namely as a space of all eigenfunctions corresponding to a given eigenvalue of the Laplace operator on a Lobachevskii plane. This definition of Ω can also be given for an arbitrary semisimple Lie group G.

We begin by giving a definition of Ω in case H contains a vector η_0 that is invariant under the operators $T(u)$, where u ranges over the maximal compact subgroup U.

With every $\xi \in H$ we associate a function on G,

$$\xi \to \varphi_\xi(g) = (T(g)\xi, \eta_0).$$

It is easy to see that the functions $\varphi_\xi(g)$ satisfy the condition

$$\varphi_\xi(ug) = \varphi_\xi(g)$$

for every $u \in U$, that is, they are constant on the cosets of U in G. So they can be regarded as functions $\varphi_\xi(x)$ on the symmetric space $S = U \setminus G$.

We consider the Laplace operators on S, that is, differential operators Δ on S that commute with the group translations:

$$(\Delta\varphi_\xi)(xg) = \Delta(\varphi_\xi(xg)).$$

From the irreducibility of the space of functions $\varphi_\xi(x)$ it follows that all these functions are eigenfunctions of Laplace operators corresponding to a fixed eigenvalue (depending only on H) for the whole set of functions.

It can be shown that our space Ω consists of all eigenfunctions of the Laplace operators on S with a fixed choice of eigenvalues.

A similar statement holds in the general case. Namely, every space Ω can be realized as a linear set of continuous vector functions $f(x)$ defined on S and subject to the following conditions:

1. Ω is closed with respect to uniform convergence on every compact set in S.
2. A representation of G acts in Ω and is defined by the following formula;

$$T(g)f(x) = \alpha(x, g)f(xg),$$

where $\alpha(x, g)$ is some continuous matrix function of x and g.

3. The original space H is contained in Ω as an everywhere dense subset. More accurately, to every $\xi \in H$ there corresponds a function $f_\xi(x) \in \Omega$, where

$$f_{T(g)\xi}(x) = T(g)f_\xi(x),$$

and if $(\xi_n, \xi_n) \to 0$, then the functions $f_{\xi_n}(x)$ tend to zero uniformly on every compact set in S.

4. The space Ω consists of all eigenfunctions of a set of differential operators (with matrix coefficients) on G.

§ 5. THE TRACE FORMULA FOR THE GROUP G OF REAL UNIMODULAR MATRICES OF ORDER 2

1. Statement of the Problem. In **§ 2.4** we obtained the trace formula for a locally compact group G and a discrete subgroup Γ for which $\Gamma \setminus G$ is a compact space.

We recall: let $\chi(\gamma)$ be a finite-dimensional representation of Γ, and $T(g)$ the unitary representation of G induced by it. Denoting by $\sigma_k(g)$ the characters of the irreducible representations occurring in the decomposition of $T(g)$ we obtained the following relation:

$$\sum_k \int \varphi(g)\sigma_k(g)\,dg = \sum_{\gamma\in\Gamma} \operatorname{Tr}\chi(\gamma)\mu(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma\setminus G} \varphi(g^{-1}\gamma g)\,dg. \quad (1)$$

Here Γ_γ and G_γ are the centralizers of γ in Γ and G, respectively; $\operatorname{Tr}\chi(\gamma)$ is the trace of the matrix $\chi(\gamma)$; γ ranges over precisely one representative from each class of conjugate elements in Γ; $\varphi(g)$ is a positive definite continuous function satisfying the following inequality

$$|\varphi(g_0)| \leq \int_U \varphi_1(g_0 g)\,dg, \quad (2)$$

where U is a compact neighborhood of the unit element, and $\varphi_1(g)$ a non-negative function summable on G.

The formula (1) contains a lot of information about how $T(g)$ splits into irreducible representations. For example, in the case of a compact group G we can easily obtain from the formula an expression for the multiplicity with which a given irreducible representation is contained in $T(g)$. For by substituting for $\varphi(g)$ the character $\sigma(g)$ of the irreducible representation we find that the multiplicity N with which this representation occurs in $T(g)$ is equal to

$$N = \frac{1}{n_\Gamma} \sum_{\gamma\in\Gamma} \overline{\sigma(\gamma)} \operatorname{Tr}\chi(\gamma)$$

(n_Γ is the order of Γ), see **§ 2.4**. Note that when the multiplicities N are known to us, then we also know which irreducible representations occur in $T(g)$ and which do not.

In this section we shall discuss the trace formula for a noncompact group—the group G of real unimodular matrices of order 2.

To determine the multiplicities N in the case of a compact group we substituted in the trace formula for $\varphi(g)$ the characters $\sigma(g)$ of the irreducible representations. Here again we pass from the functions $\varphi(g)$ to the functions

$$h(\sigma) = \int \varphi(g)\sigma(g)\,dg,$$

where σ ranges over the characters of the irreducible representations.

The left-hand side of (1) permits passage to $h(\sigma)$ in an obvious fashion. The main problem consists in passing to $h(\sigma)$ on the right-hand side of the equation. This problem will be solved in **§ 5.3**

through **5.6**. The final relation for the functions $h(\sigma)$ will be obtained in **§ 5.6**. From this relation we draw important conclusions in **§ 5.7** and **§ 5.10**, namely:

1. Formulae for the multiplicities with which the representations of the discrete series occur in $T(g)$.

2. An asymptotic formula for the number of representations $T_s(g)$ of the continuous series with the index $|s| \leq \alpha$ that occur in $T(g)$ (as $\alpha \to \infty$).

Notwithstanding the somewhat lengthy computations to be given, we emphasize that they are all elementary, require no new ideas or arguments, and follow the standard workings in these branches of representation theory.

2. The Function h. Here we give without proof the formulae for the characters of the irreducible representations of Γ. (These formulae will be derived in Chapter II, **§ 4**.)

1. The character of the representation T_s^+, $s = i\rho$, of the first principal series is concentrated on the set of hyperbolic elements (that is, on the set of matrices with real eigenvalues $\lambda \neq \pm 1$) and is given on this set by the following formula:

$$\sigma_\rho^+(g) = \frac{|\lambda_g|^{i\rho} + |\lambda_g|^{-i\rho}}{|\lambda_g - \lambda_g^{-1}|}, \tag{1}$$

where λ_g is an eigenvalue of the matrix g.

2. The character of the representation T_s^-, $s = i\rho$, of the second principal series is also concentrated on the set of hyperbolic elements and is given on this set by the following formula:

$$\sigma_\rho^-(g) = \frac{|\lambda_g|^{i\rho} + |\lambda_g|^{-i\rho}}{|\lambda_g - \lambda_g^{-1}|} \operatorname{sign} \lambda_g. \tag{2}$$

3. The character of the representation T_s, $s = \rho$, of the supplementary series is concentrated on the set of hyperbolic elements and is given on this set by the following formula:

$$\sigma_{i\rho}(g) = \frac{|\lambda_g|^{\rho} + |\lambda_g|^{-\rho}}{|\lambda_g - \lambda_g^{-1}|}, \qquad 0 < \rho < 1. \tag{3}$$

(Therefore, this formula is obtained from the formula for the character of the first principal series by analytic continuation with respect to ρ.)

4. The character of the representations of the first half of the discrete series is given by the following formulae: On the set of hyperbolic elements

$$\sigma_n^+(g) = \frac{\lambda_g^{-n}}{\lambda_g - \lambda_g^{-1}}, \tag{4}$$

where λ_g is the eigenvalue of g of greatest modulus. On the set of elliptic elements

$$\sigma_n^+(g) = \frac{e^{-in\varphi}}{e^{i\varphi} - e^{-i\varphi}}, \tag{5}$$

where φ is the angle of rotation corresponding to the matrix g. $\left(\text{In other words, } g \text{ is conjugate to the matrix } \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.\right)$

5. The character of the representations of the second half of the discrete series is given by the following formulae: On the set of hyperbolic elements

$$\sigma_n^-(g) = \sigma_n^+(g) = \frac{\lambda_g^{-n}}{\lambda_g - \lambda_g^{-1}}. \tag{6}$$

On the set of elliptic elements

$$\sigma_n^-(g) = \frac{e^{in\varphi}}{e^{-i\varphi} - e^{i\varphi}}, \tag{7}$$

where φ is defined as in (5).

Now we associate the following collection of functions with the functions $\varphi(g)$ on G that satisfy the estimate (2) of **§ 5.1**:

$$\left.\begin{aligned} h^+(\rho) &= \int \varphi(g)\sigma_\rho^+(g)\, dg, \\ h^-(\rho) &= \int \varphi(g)\sigma_\rho^+(g)\, dg, \\ h^+(i\rho) &= \int \varphi(g)\sigma_{i\rho}^+(g)\, dg, \\ h_n^+ &= \int \varphi(g)\sigma_n^+(g)\, dg, \\ h_n^- &= \int \varphi(g)\sigma_n^-(g \end{aligned}\right\}^{\dagger} \tag{8}$$

Our task is to pass to the trace formula (1) of **§ 5.1** from the functions $\varphi(g)$ to the functions h.

Obviously the left-hand side of (1) can be written in the following form:

$$\sum h^+(\rho_k) + \sum h^-(\rho_l) + \sum h^+(i\rho_m) + \sum h_n^+ + \sum h_{n_t}^-,$$

where the sum is taken over those indices ρ_k, n_s, n_t of the representations of the principal, supplementary, and discrete series that occur in $T(g)$. Thus, our main task is to express the right-hand side of the trace formula (1) of **§ 5.1** in terms of the functions h:

$$I = \sum_{\gamma\in\Gamma} \operatorname{Tr} \chi(\gamma)\mu(\Gamma_\gamma G_\gamma) \int_{G_\gamma G} \varphi(g^{-1}\gamma g)\, dg \tag{9}$$

† In other words, $h^+(\rho) = \operatorname{Tr}\,[\int \varphi(g) T_{i\rho}^+(g)\, dg]$.

We also assume that Γ contains the matrix $-e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (The case when $-e$ is not contained in Γ will be discussed separately in **§ 5.11.**)

Since $\Gamma \setminus G$ is compact, Γ does not contain parabolic elements; thus, the elements $\gamma \neq \pm e$ of Γ are hyperbolic or elliptic.

We shall find the contribution to the trace formula separately for hyperbolic elements, elliptic elements, and the elements e, $-e$.

3. Contribution of the Hyperbolic Elements to the Trace Formula. Let $I(\delta)$ be the integral of $\varphi(g)$ with respect to the class of elements conjugate to the diagonal matrix $\delta = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \neq \pm 1$:

$$I(\delta) = \int\limits_{D-G} f(g^{-1}\,\delta g)\, d\tilde{g}. \tag{1}$$

(The integration is taken over the space of conjugacy classes of the subgroup D of diagonal matrices in G.)

The following formula expresses $I(\delta)$ in terms of $h(\rho)$:

$$I(\delta) = \frac{1}{4\pi\,|\lambda - \lambda^{-1}|} \int\limits_{-\infty}^{+\infty} (h^{+}(\rho) + h^{-}(\rho) \operatorname{sign} \lambda)\, |\lambda|^{i\rho}\, d\rho. \tag{2}$$

In order not to interrupt the exposition we shall derive the formula (2) at the end of this section. On the basis of (2) we can now find the contribution of the hyperbolic elements to the trace formula.

We call an hyperbolic element $\gamma \in \Gamma$ *primitive* if the following two conditions hold:

1. γ is not a power of any other element in Γ.
2. The eigenvalues of γ are positive.

The corresponding class $\{\gamma\}$ of conjugate elements is also called primitive. Clearly, every hyperbolic element γ can be represented in the form $\gamma = \pm\gamma_1^k$, where γ_1 is a primitive element.

Let γ be a primitive hyperbolic element. Then it is easy to see that the classes $\{\gamma^k\}$ and $\{-\gamma^l\}$, k, $l = 1, 2, \ldots$ are all pairwise distinct. Furthermore, if γ_1 is another primitive element with positive eigenvalues such that $\{\gamma_1\} \neq \{\gamma\}$, then the classes $\{\gamma_1^{k_1}\}$, $\{-\gamma_1^{l_1}\}$ and $\{\gamma^k\}$, $\{-\gamma^l\}$ are all distinct.

On the right-hand side of the trace formula we examine the set of terms corresponding to the classes $\{\gamma^k\}$ and $\{-\gamma^l\}$, where γ is a primitive hyperbolic element.

Since $G_{\gamma^k} = G_{-\gamma^l} = G_\gamma$, $\Gamma_{\gamma^k} = \Gamma_{-\gamma^l} = \Gamma_\gamma$, the corresponding sum has the following form:

$$\mu(\Gamma_\gamma \setminus G_\gamma) \sum_{k=1}^{\infty} \operatorname{Tr} \chi(\gamma^k) \int_{G_\gamma \setminus G} \varphi(g^{-1}\gamma^k g)\, dg$$
$$+ \mu(\Gamma_\gamma \setminus G_\gamma) \sum_{k=1}^{\infty} \operatorname{Tr} \chi(-\gamma^k) \int_{G_\gamma \setminus G} \varphi(-g^{-1}\gamma^k g)\, dg. \quad (3)$$

Without loss of generality we may assume that γ is a diagonal matrix

$$\gamma = \begin{pmatrix} \lambda_\gamma & 0 \\ 0 & \lambda_\gamma^{-1} \end{pmatrix},$$

where $\lambda_\gamma > 1$. Then G_γ coincides with the group D of all diagonal matrices, and Γ_γ is the subgroup generated by the matrices γ and $-\gamma$.

We calculate $\mu(\Gamma_\gamma \setminus G_\gamma)$. The invariant measure on the subgroup $G_\gamma = D$ of diagonal matrices $\delta = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ is given by the following formula:

$$d\delta = \frac{d\lambda}{|\lambda|}$$

(here it is assumed that the measure $d\tilde{g}$ on $G_\gamma \setminus G$ is normed so that $dg = d\delta\, d\tilde{g}$ when $g = \delta\tilde{g}$). Since the fundamental domain in G_γ relative to Γ_γ consists of the matrices $\delta = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $1 < \lambda < \lambda_\gamma$, we have

$$\mu(\Gamma_\gamma \setminus G_\gamma) = \int_1^{\lambda_\gamma} \frac{d\lambda}{\lambda} = \ln \lambda_\gamma.$$

In (3) we substitute for the integrals their expressions in terms of $h(\rho)$ in accordance with (2). As a result, after regrouping the terms, we obtain the following expression:

$$\sum_{s=1}^{\infty} \frac{\ln \lambda_\gamma \operatorname{Tr} \chi(\gamma^s)}{2\pi\, |\lambda_\gamma^s - \lambda_\gamma^{-s}|}$$
$$\left(\frac{1+\varepsilon}{2} \int_{-\infty}^{+\infty} h^+(\rho)\, \lambda_\gamma^{i\rho s}\, d\rho + \frac{1-\varepsilon}{2} \int_{-\infty}^{+\infty} h^-(\rho)\, \lambda_\gamma^{i\rho s}\, d\rho \right),$$

where λ_γ is the largest eigenvalue of γ; $\varepsilon = 1$ when $\chi(-e) = E$ $\varepsilon = -1$ when $\chi(-e) = -E$, where E is the identity operator.

To find the contribution to the trace formula of all hyperboli elements, we sum the expression so obtained over the set of al primitive classes of hyperbolic elements. The final result is:

The contribution of the hyperbolic elements γ to the trace formula is

$$\sum_{\{\gamma\}} \sum_{s=1}^{\infty} \frac{\ln \lambda_\gamma \operatorname{Tr} \chi(\gamma^s)}{2\pi\, |\lambda_\gamma^s - \lambda_\gamma^{-s}|}$$
$$\left(\frac{1+\varepsilon}{2} \int_{-\infty}^{+\infty} h^+(\rho)\, \lambda_\gamma^{i\rho s}\, d\rho + \frac{1-\varepsilon}{2} \int_{-\infty}^{+\infty} h^-(\rho)\, \lambda_\gamma^{i\rho s}\, d\rho \right), \quad (4)$$

where the sum is taken over the set of all primitive classes $\{\gamma\}$ of conjugate hyperbolic elements; λ_γ is the largest eigenvalue of γ; $\varepsilon = 1$ when $\chi(-e) = E$, and $\varepsilon = -1$ when $\chi(-e) = -E$, where E is the unit operator.

Now we derive formula (2). This derivation is based on the following integral relation for the function $I(\delta)$, which we give here without proof:

$$\int_{|\lambda|<1} I(\delta)\omega(\delta)\, d\delta = \int_{|\lambda|>1} I(\delta)\omega(\delta)\, d\delta = \int_{G_{hyp}} \varphi(g)\omega(g)\, |\lambda_g - \lambda_g^{-1}|^{-2}\, dg, \quad (5)$$

where the integral on the right is taken over the set G_{hyp} of hyperbolic elements of G, λ_g, λ_g^{-1} are the eigenvalues of g; $\omega(g)$ is an arbitrary function on G that is constant on each class of conjugate elements.

From this integral relation it follows that for every real ρ the following equation holds:

$$\int_{-\infty}^{+\infty} I(\delta)\, |\lambda - \lambda^{-1}|\, |\lambda|^{i\rho-1}\, d\lambda = \int_{G_{hyp}} \varphi(g)\, \frac{|\lambda_g|^{i\rho} + |\lambda_g|^{-i\rho}}{|\lambda_g - \lambda_g^{-1}|}\, dg = h^+(\rho).$$

Similarly,

$$\int_{-\infty}^{+\infty} I(\delta)\, |\lambda - \lambda^{-1}|\, |\lambda|^{i\rho-1} \operatorname{sign} \lambda\, d\lambda = h^-(\rho).$$

Hence, we obtain

$$\int_0^{\infty} I(\delta)\, |\lambda - \lambda^{-1}|\, \lambda^{i\rho-1}\, d\lambda = \tfrac{1}{2}(h^+(\rho) + h^-(\rho)).$$

Consequently, by the formula for the inverse Mellin transform

$$I(\delta) = \frac{1}{4\pi\, |\lambda - \lambda^{-1}|} \int_{-\infty}^{+\infty} (h^+(\rho) + h^-(\rho))\, \lambda^{i\rho}\, d\rho$$

for $\lambda > 0$. Similarly,

$$I(\delta) = \frac{1}{4\pi\, |\lambda - \lambda^{-1}|} \int_{-\infty}^{+\infty} (h^+(\rho) - h^-(\rho))\, |\lambda|^{i\rho}\, d\rho$$

for $\lambda > 0$.

4. Contribution of the Elliptic Elements. Let $I(u)$ be the integral of $\varphi(g)$ over the class of elements conjugate to the orthogonal matrix $u = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, $0 < |t| < \pi$:

$$I(u) = \int_{U\backslash G} \varphi(g^{-1}ug)\, d\tilde{g}. \tag{1}$$

(The integration is taken over the set of conjugacy classes of the subgroup U of orthogonal matrices in G.)

The following formula expresses $I(u)$ in terms of $h(\rho)$ and h_n:

$$I(u) = -\frac{1}{4\pi i \sin t}\left\{\frac{h_0^+ - h_0^-}{2} + \sum_{n=1}^{\infty} (h_n^+ e^{int} - h_n^- e^{-int}\right\} + \frac{1}{16\pi \sin |t|}\int_{-\infty}^{+\infty}$$

$$\left(h^+(\rho)\frac{\cosh \frac{(2\,|t| - \pi)\rho}{2}}{\cosh \frac{\pi\rho}{2}} - h^-(\rho)\frac{\sinh \frac{(2\,|t| - \pi)\rho}{2}}{\sinh \frac{\pi\rho}{2}}\right) d\rho. \tag{2}$$

As in the preceding subsection, in order not to interrupt the exposition, we shall derive formula (2) at the end of this subsection. On the basis of (2) we can now find the contribution of the elliptic elements to the trace formula.

Observe, first of all, that all elliptic elements $\gamma \in \Gamma$ are of finite order. For if there were an elliptic element γ of infinite order, then we could select from the sequence $\gamma, \gamma^2, \ldots, \gamma^n, \ldots$ a convergent subsequence of pairwise distinct elements, and this contradicts the fact that Γ is discrete.

We call an elliptic element $\gamma \in \Gamma$ *primitive* if the following conditions hold:

1. γ is not a power of another element of Γ of higher order.
2. Among the elements $\gamma, \gamma^2, \ldots$ γ yields the rotation by the smallest positive angle.

The corresponding class of conjugate elements $\{\gamma\}$ is also called primitive.

Clearly every elliptic element is the power of a primitive elliptic element.

We note that the order of a primitive element is always even. For suppose that an elliptic element γ is of odd order k, $\gamma^k = e$. Obviously, then $-\gamma$ is of order $2k$, and $(-\gamma)^{k+1} = \gamma$.

Let γ be a primitive elliptic element of order $2k$.

From the definition it follows that γ is conjugate in G to the matrix $\begin{pmatrix} \cos\frac{\pi}{k} & \sin\frac{\pi}{k} \\ -\sin\frac{\pi}{k} & \cos\frac{\pi}{k} \end{pmatrix}$. Without loss of generality, we may assume that

$$\gamma = \begin{pmatrix} \cos\frac{\pi}{k} & \sin\frac{\pi}{k} \\ -\sin\frac{\pi}{k} & \cos\frac{\pi}{k} \end{pmatrix}.$$

We examine the classes of conjugate elements $\{\gamma^s\}$, $\{-\gamma^t\} = \{\gamma^{k+t}\}$, $s, t = 1, \ldots, k-1$. It is clear that all these classes are distinct. We say that these classes are *connected with a given primitive element* γ. It is easy to verify that the classes connected with two primitive elliptic elements γ and γ_1 that are not conjugate in Γ are all distinct.

In the trace formula we consider the set of terms connected with a given primitive elliptic element γ, that is, the terms corresponding to the classes $\{\gamma^s\}$ and $\{-\gamma^t\}$, $s, t = 1, \ldots, k-1$.

The corresponding sum has the form

$$\mu(\Gamma_\gamma \setminus G_\gamma) \sum_{s=1}^{k-1} \operatorname{Tr} \chi(\gamma^s) \int_{G_\gamma \setminus G} \varphi(g^{-1}\gamma^s g)\, dg$$
$$+ \mu(\Gamma_\gamma \setminus G_\gamma) \sum_{s=1}^{k-1} \operatorname{Tr} \chi(-\gamma^s) \int_{G_\gamma \setminus G} \varphi(-g^{-1}\gamma^s g)\, dg. \quad (3)$$

Note that in this case, G_γ is the group of all orthogonal matrices $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$; Γ_γ its cyclic subgroup of order $2k$ generated by γ. Consequently, if the measure on G_γ is normed by the condition

$$\mu(G_\gamma) = 2\pi,$$

then

$$\mu(\Gamma_\gamma \setminus G_\gamma) = \frac{\pi}{k}.$$

In (3) we substitute for the integrals their expressions in terms of $h(\rho)$ and h_n in accordance with (2). As a result, after regrouping terms, we find that the contribution to the trace formula of classes connected with a given primitive elliptic element γ of order $2k$ is

equal to

$$
\begin{aligned}
&-\frac{1}{4ki}\sum_{s=1}^{k-1}\frac{\operatorname{Tr}\chi(\gamma^s)}{\sin\frac{\pi s}{k}}\left\{(1-\varepsilon)\frac{h_0^+-h_0^-}{2}\right.\\
&\qquad\left.+\sum_{n=1}^{\infty}(1-(-1)^n\varepsilon)\left(h_n^+e^{i(\pi sn/k)}-h_n^-e^{-i(\pi sn/k)}\right)\right\}\\
&+\frac{1}{16k}\sum_{s=1}^{k-1}\frac{\operatorname{Tr}\chi(\gamma^s)}{\sin\frac{\pi s}{k}}\int_{-\infty}^{+\infty}\left[(1+\varepsilon)h^+(\rho)\frac{\cosh\frac{\pi(2s-k)\rho}{2k}}{\cosh\frac{\pi\rho}{2}}\right.\\
&\qquad\left.-(1-\varepsilon)h^-(\rho)\frac{\sinh\frac{\pi(2s-k)\rho}{2k}}{\sinh\frac{\pi\rho}{2}}\right]d\rho.
\end{aligned}
$$

To find the contribution to the trace formula of all elliptic elements we sum the expressions over the set of all primitive classes of elliptic elements.

As the final result we have:

The contribution to the trace formula of the elliptic elements is

$$
\begin{aligned}
&-\sum_{\{\gamma\}}\sum_{s=1}^{k-1}\frac{1}{4ki}\frac{\operatorname{Tr}\chi(\gamma^s)}{\sin\frac{\pi s}{k}}\left\{(1-\varepsilon)\frac{h_0^+-h_0^-}{2}+\sum_{n=1}^{\infty}(1-(-1)^n\varepsilon)\right.\\
&\qquad\left.\left(h_n^+e^{i(\pi sn/k)}-h_n^-e^{-i(\pi sn/k)}\right)\right\}+\sum_{\{\gamma\}}\sum_{s=1}^{k-1}\frac{1}{16k}\frac{\operatorname{Tr}\chi(\gamma^s)}{\sin\frac{\pi s}{k}}\\
&\qquad\int_{-\infty}^{+\infty}\left[(1+\varepsilon)h^+(\rho)\frac{\cosh\frac{(2s-k)\pi\rho}{2k}}{\cosh\frac{\pi\rho}{2}}\right.\\
&\qquad\qquad\left.-(1-\varepsilon)h^-(\rho)\frac{\sinh\frac{(2s-k)\pi\rho}{2k}}{\sinh\frac{\pi\rho}{2}}\right]d\rho, \qquad (4)
\end{aligned}
$$

where the summation is taken over the set of all primitive classes $\{\gamma\}$ *of elliptic elements, and* $2k$ *is the order of the primitive elliptic element* γ.

Now we derive formula (2). This derivation is based on the following integral relation for the function $I(u)$ which we give here without proof:

$$\int_{-\pi}^{\pi} I(u)\omega(u)\,dt = \int_{G_{el}} \varphi(g)\,|e^{it} - e^{-it}|^{-2}\,\omega(g)\,dg, \tag{5}$$

where $u = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, the integral on the right is taken over the set G_{el} of elliptic elements of G; e^{it}, e^{-it} are the eigenvalues of g; $\omega(g)$ is an arbitrary function on G that is constant on each class of conjugate elements.

From this relation it follows that

$$\int_{-\pi}^{\pi} I(u)(e^{-it} - e^{it})e^{-int}\,dt = \int_{G_{el}} \varphi(g)\,\frac{e^{-int}}{e^{it} - e^{-it}}\,dg.$$

We compare the expressions so obtained with the formulae defining h_n^+ and h_n^-:

$$h_n^+ = \int_{G_{el}} \varphi(g)\,\frac{e^{int}}{e^{it} - e^{-it}}\,dg + \int_{G_{hyp}} \varphi(g)\,\frac{\lambda^{-n}}{\lambda - \lambda^{-1}}\,dg,$$

$$h_n^- = -\int_{G_{el}} \varphi(g)\,\frac{e^{-int}}{e^{it} - e^{-it}}\,dg + \int_{G_{hyp}} \varphi(g)\,\frac{\lambda^{-n}}{\lambda - \lambda^{-1}}\,dg,$$

where the first integrals are taken over the set of elliptic elements and the second over the set of hyperbolic elements g, and λ is the eigenvalue of greatest modulus of g; $n \geq 0$.

We find that

$$\int_{-\pi}^{\pi} I(u)(e^{-it} - e^{it})e^{-int}\,dt = h_n^+ - \int_{G_{hyp}} \varphi(g)\,\frac{\lambda^{-n}}{\lambda - \lambda^{-1}}\,dg,$$

$$\int_{-\pi}^{\pi} I(u)(e^{-it} - e^{it})e^{int}\,dt = -h_n^- + \int_{G_{hyp}} \varphi(g)\,\frac{\lambda^{-n}}{\lambda - \lambda^{-1}}\,dg.$$

On the basis of the formula for the inverse Fourier transform we find that

$$I(u) = \frac{1}{2\pi(e^{-it} - e^{it})}\left\{\frac{h_0^+ - h_0^-}{2} + \sum_{n=1}^{\infty}(h_n^+e^{int} - h_n^-e^{-int}) - \sum_{n=1}^{\infty}\int_{G_{hyp}} \varphi(g)\,\frac{\lambda^{-n}(e^{int} - e^{-int})}{\lambda - \lambda^{-1}}\,dg\right\}. \tag{6}$$

We now express the last term in this formula in terms of $h^+(\rho)$ and $h^-(\rho)$.

First of all, we note that

$$\sum_{n=1}^{\infty} \lambda^{-n}(e^{int} - e^{-int}) = \frac{\lambda(e^{it} - e^{-it})}{1 - 2\lambda\cos t + \lambda^2}.$$

Consequently,

$$I_t = -\frac{1}{2\pi(e^{-it}-e^{it})}\sum_{n=1}^{\infty}\int_{G_{hyp}}\varphi(g)\,\frac{\lambda^{-n}(e^{int}-e^{-int})}{\lambda-\lambda^{-1}}\,dg$$

$$-\frac{1}{2\pi}\int_{G_{hyp}}\varphi(g)\,\frac{\lambda}{(\lambda-\lambda^{-1})(1-2\lambda\cos t+\lambda^2)}\,dg.$$

On the basis of the formulae derived at the end of the preceding subsection we obtain

$$I_t = \frac{1}{8\pi^2}\int_1^{\infty}\int_{-\infty}^{+\infty}(h^+(\rho)+h^-(\rho))\,\frac{\lambda^{i\rho}}{1-2\lambda\cos t+\lambda^2}\,d\rho\,d\lambda$$

$$+\frac{1}{8\pi^2}\int_1^{\infty}\int_{-\infty}^{+\infty}(h^+(\rho)-h^-(\rho))\,\frac{\lambda^{i\rho}}{1+2\lambda\cos t+\lambda^2}\,d\rho\,d\lambda.$$

We simplify this expression. First of all, using the fact that $h^+(-\rho) = h^+(\rho)$ and $h^-(-\rho) = h^-(\rho)$ we can rewrite it in the following form:

$$I_t = \frac{1}{16\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}(h^+(\rho)+h^-(\rho))\,\frac{\lambda^{i\rho}}{1-2\lambda\cos t+\lambda^2}\,d\lambda\,d\rho$$

$$+\frac{1}{16\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}(h^+(\rho)-h^-(\rho))\,\frac{\lambda^{i\rho}}{1+2\lambda\cos t+\lambda^2}\,d\lambda\,d\rho.$$

Now we integrate over λ by using the following standard formulae:

$$\int_0^{\infty}\frac{\lambda^{i\rho}}{1+2\lambda\cos t+\lambda^2}\,d\lambda = \frac{\pi\sinh\rho t}{\sinh\rho\pi\sin t}, \qquad 0<|t|<\pi,$$

$$\int_0^{\infty}\frac{\lambda^{i\rho}}{1-2\lambda\cos t+\lambda^2}\,d\lambda = -\frac{\pi\sinh\rho(t-\pi)}{\sinh\rho\pi\sin t}, \qquad 0<t<\pi,$$

$$\int_0^{\infty}\frac{\lambda^{i\rho}}{1-2\lambda\cos t+\lambda^2}\,d\lambda = -\frac{\pi\sinh\rho(t+\pi)}{\sinh\rho\pi\sin t}, \qquad -\pi<t<0.$$

As a result we find that

$$I_t = I_{-t} = -\frac{1}{16\pi\sin t}\left\{\int_{-\infty}^{+\infty}(h^+(\rho)+h^-(\rho))\,\frac{\sinh\rho(t-\pi)}{\sinh\rho\pi}\,d\rho\right.$$

$$\left.-\int_{-\infty}^{+\infty}(h^+(\rho)-h^-(\rho))\,\frac{\sinh\rho t}{\sinh\rho\pi}\,d\rho\right\},$$

where $0 < t < \pi$. After regrouping terms we have

$$I_t = I_{-t} = \frac{1}{16\pi \sin t} \int_{-\infty}^{+\infty} \left(h^+(\rho) \frac{\cosh \frac{(2t-\pi)\rho}{2}}{\cosh \frac{\pi\rho}{2}} - h^-(\rho) \frac{\sinh \frac{(2t-\pi)\rho}{2}}{\sinh \frac{\pi\rho}{2}} \right) d\rho,$$

where $0 < t < \pi$.

Substituting this expression in the formula (6) for $I(u)$ we obtain the required formula:

$$I(u) = -\frac{1}{4\pi i \sin t} \left\{ \frac{h_0^+ - h_0^-}{2} + \sum_{n=1}^{\infty} (h_n^+ e^{int} - h_n^- e^{-int}) \right\}$$

$$+ \frac{1}{16\pi \sin |t|} \int_{-\infty}^{+\infty} \left(h^+(\rho) \frac{\cosh \frac{(2|t|-\pi)\rho}{2}}{\cosh \frac{\pi\rho}{2}} - h^-(\rho) \frac{\sinh \frac{(2|t|-\pi)\rho}{2}}{\sinh \frac{\pi\rho}{2}} \right) d\rho,$$

where $|t| < \pi$.

5. Contribution of the Elements e and $-e$ to the Trace Formula. Clearly the terms corresponding to $\gamma = \pm e$ in the trace formula are

$$\nu\mu(\Gamma \setminus G)[\varphi(e) + \varepsilon\varphi(-e)] \tag{1}$$

where ν is the dimension of the representation $\chi(\gamma)$; where $\mu(\Gamma \setminus G)$ is the volume of $\Gamma \setminus G$, $\varepsilon = 1$ when $\chi(-e) = E$ and $\varepsilon = -1$ when $\chi(-e) = -E$, where E is the unit operator. So we have to express $\varphi(e)$ and $\varphi(-e)$ in terms of the function h. These expressions will be obtained in Chapter 2, **§ 6**. We quote them here without proof.

The following formulae hold:

$$\varphi(e) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \left((h^+(\rho)\rho \tanh \frac{\pi\rho}{2} + h^-(\rho)\rho \coth \frac{\pi\rho}{2} \right) d\rho + \frac{1}{\pi^2} \sum_{n=1}^{\infty} n(h_n^+ + h_n^-); \tag{2}$$

$$\varphi(-e) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \left(h^+(\rho)\rho \tanh \frac{\pi\rho}{2} - h^-(\rho)\rho \coth \frac{\pi\rho}{2} \right) d\rho + \frac{1}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n-1} n(h_n^+ + h_n^-). \tag{2'}$$

Thus, *the contribution to the trace formula of the elements e and $-e$ is*

$$\nu\mu(\Gamma \setminus G) \left\{ \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \left(h^+(\rho)(1 + \varepsilon)\rho \tanh \frac{\pi\rho}{2} + h^-(\rho)(1 - \varepsilon)\rho \coth \frac{\pi\rho}{2} \right) d\rho \right.$$

6. The Final Trace Formula. We have now found the separate contributions to the trace formula of the hyperbolic elements, the elliptic elements, and the elements e and $-e$. Let us note the final trace formula.

Let $h^+(\rho)$, $h^-(\rho)$, $h^+(i\rho)$, h_n^+, h_n^- be the Fourier transforms of the functions $\varphi(g)$ corresponding to the various series of irreducible unitary representations of G (see formula (8) in **§ 5.2**).

Suppose that Γ is a discrete subgroup of G such that Γ contains the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and that the space $\Gamma \setminus G$ is compact, that $\chi(\gamma)$ is a finite-dimensional unitary representation of Γ and that $T(g)$ is the representation of G induced by $\chi(\gamma)$.

Then we have the following trace formula:

$$\sum h^+(\rho_k) + \sum h^+(\rho_l) + \sum h^+(i\rho_m) + \sum h_{n_i}^+ + \sum h_{n_j}^-$$

$$= \frac{1}{4\pi^2}\, \nu\mu(\Gamma \quad G) \int_{-\infty}^{+\infty} \Big[h^+(\rho)(1+\varepsilon)\rho \tanh \frac{\pi\rho}{2}$$

$$+ h^-(\rho)(1-\varepsilon)\rho \coth \frac{\pi\rho}{2}\Big]\, d\rho$$

$$+ \frac{1}{\pi^2}\, \nu\mu(\Gamma \quad G) \sum_{n=1}^{\infty} n[1 + (-1)^{n-1}\varepsilon](h_n^+ + h_n^-)$$

$$+ \sum_{\{\gamma\}} \sum_{k=1}^{\infty} \frac{\ln \lambda_\gamma \operatorname{Tr} \chi(\gamma^k)}{2\pi\, |\lambda_\gamma^k - \lambda_\gamma^{-k}|} \left(\frac{1+\varepsilon}{2} \int_{-\infty}^{+\infty} h^+(\rho)\, \lambda_\gamma^{i\rho k}\, d\rho \right.$$

$$\left. + \frac{1-\varepsilon}{2} \int_{-\infty}^{+\infty} h^-(\rho)\, \lambda_\gamma^{i\rho k}\, d\rho \right)$$

$$- \sum_{\{\gamma\}} \sum_{s=1}^{k-1} \frac{1}{4ki} \frac{\operatorname{Tr} \chi(\gamma^s)}{\sin \dfrac{\pi s}{k}} \left\{ \frac{1-\varepsilon}{2} (h_0^+ - h_0^-) \right.$$

$$\left. + \sum_{n=1}^{\infty} [1 - (-1)^n \varepsilon] h_n^+ e^{-i(\pi s n/k)} - h_n^- e^{-i(\pi s n/k)} \right\}$$

$$+ \sum_{\{\gamma\}} \sum_{s=1}^{k-1} \frac{1}{16k} \frac{\operatorname{Tr} \chi(\gamma^s)}{\sin \dfrac{\pi s}{k}} \int_{-\infty}^{+\infty} \left[h^+(\rho)(1+\varepsilon) \frac{\cosh \dfrac{(2s-k)\pi\rho}{2k}}{\cosh \dfrac{\pi\rho}{2}} \right.$$

$$\left. - h^-(\rho)(1-\varepsilon) \frac{\sinh \dfrac{(2s-k)\pi\rho}{2k}}{\sinh \dfrac{\pi\rho}{2}} \right] d\rho.$$

On the left-hand side of this formula the sum is taken over the representations that occur in the decomposition of $T(g)$; each term is counted with the appropriate multiplicity.

On the right-hand side the first two terms are the contribution of the elements e and $-e$; the third term is the contribution of the hyperbolic elements; the last two terms are the contributions of the elliptic elements.

We recapitulate our notation: ν is the dimension of $\chi(\gamma)$; $\mu(\Gamma \setminus G)$ is the volume of the fundamental domain; $\varepsilon = 1$ when $\chi(-e) = E$, $\varepsilon = -1$ when $\chi(-e) = -E$, where E is the unit operator; $\operatorname{Tr} \chi(\gamma)$ is the trace of the matrix $\chi(\gamma)$; λ_γ is the largest eigenvalue of the primitive hyperbolic element to γ; $2k$ is the order of the primitive elliptic element γ. The summation in the third sum is taken over the set of primitive classes $\{\gamma\}$ of conjugate hyperbolic elements; in the fourth and fifth sum, over the set of primitive classes $\{\gamma\}$ of conjugate elliptic elements.

7. Formulae for the Multiplicities of the Representations of the Discrete Series. On the basis of the trace formula we obtain at once the formulae for the multiplicity with which a representation of the discrete series occurs in $T(g)$.

Our arguments are based on the following fact. We can construct continuous positive definite functions $\varphi_n^+(g)$, $\varphi_n^-(g)$, $n = 1, 2, \ldots$ on G that satisfy the following conditions:

1. $T_{\varphi_n^+} = \int \varphi_n^+(g) T(g)\, dg$ and $T_{\varphi_n^-} = \int \varphi_n^-(g) T(g)\, dg$ are completely continuous integral operators.

From this condition it follows that the trace formula is applicable to the functions $\varphi_n^+(g)$ and $\varphi_n^-(g)$.

2. For the functions $\varphi_n^+(g)$ we have $h^+(\rho) = h^-(\rho) = 0$, $h_i^- = 0$, $i = 0, 1, 2, \ldots$; $h_j^+ = 0$ for $j \neq n$; $h_n^+ \neq 0$. Similarly, for the functions $\varphi_n^-(g)$ we have $h^+(\rho) = h^-(\rho) = 0$, $h_i^+ = 0$, $i = 0, 1, 2, \ldots$; $h_j^- = 0$ for $j \neq n$; $h_n^- \neq 0$.

In order not to interrupt the exposition we shall construct these functions later, in **§ 5.9.**

Let us apply the trace formula to the functions $\varphi_n^+(g)$. By condition 2, the only nonzero terms in this formula are those with h_n^+. As a result, after cancelling h_n^+ we obtain the formula for the multiplicity with which the representation $T_n^+(g)$ of the discrete series occurs in $T(g)$.

The multiplicity N_n^+ with which the representation $T_n^+(g)$, $n > 0$, of the discrete series occurs in $T(g)$ is expressed by the following formula:

$$N_n^+ = [1 + (-1)^{n-1}\varepsilon] \left\{ \frac{\nu\mu(\Gamma \setminus G)}{\pi^2}\, n - \sum_{\{\gamma\}} \sum_{s=1}^{k-1} \frac{1}{4ki} \frac{\operatorname{Tr} \chi(\gamma^s)}{\sin \dfrac{\pi s}{k}}\, e^{i\pi s n/k} \right\}. \tag{1}$$

Similarly by applying the trace formula to the functions $\varphi_n^-(g)$ we obtain:

The multiplicity N_n^- with which the representation $T_n^-(g)$, $n > 0$, of the second half of the discrete series occurs in $T(g)$ is expressed by the following formula:

$$N_n^- = [1 + (-1)^{n-1}\varepsilon]\left\{\frac{\nu\mu(\Gamma\setminus G)}{\pi^2}n + \sum_{\{\gamma\}}\sum_{s=1}^{k-1}\frac{1}{4ki}\frac{\operatorname{Tr}\chi(\gamma^s)}{\sin\frac{\pi s}{k}}e^{-i\pi sn/k}\right\}. \tag{1'}$$

In the particular case when Γ does not contain elliptic elements we have

$$N_n^+ = N_n^- = [1 + (-1)^{n-1}\varepsilon]\frac{\nu\mu(\Gamma\setminus G)}{\pi^2}n. \tag{2}$$

8. Complete Splitting of the Trace Formula. We apply the trace formula to an arbitrary function $\varphi(g)$. From the result in **§ 5.7** it follows that the terms with $h_n^\pm$ on the left- and right-hand sides of the formula are identical: consequently all these terms can be discarded. As a result we obtain a relation containing only the functions $h^+(\rho)$ and $h^-(\rho)$. We show that this relation, in its turn, can be split into a relation for $h^+(\rho)$ and one for $h^-(\rho)$.

Let us replace $\varphi(g)$ by $\varphi^+(g) = \dfrac{\varphi(g) + \varphi(-g)}{2}$. Obviously for $\varphi^+(g)$ and $\varphi(g)$ the functions $h^+(\rho)$ coincide; on the other hand, for the functions $\varphi^+(g)$ we have $h^-(\rho) = 0$. Consequently, the transition in the trace formula from $\varphi(g)$ to $\varphi^+(g)$ reduces to discarding the terms with $h^-(\rho)$ in this formula. Thus, finally we can separate the trace formula into two relations—one for the functions $h^+(\rho)$ and the other for the functions $h^-(\rho)$. These relations have the following form:

$$\sum_k h^+(\rho_k) + \sum_l h^+(i\rho_l) = \frac{1}{4\pi^2}\nu\mu(\Gamma\setminus G)(1+\varepsilon)\int_{-\infty}^{+\infty} h^+(\rho)\rho\tanh\frac{\pi\rho}{2}\,d\rho$$

$$+ (1+\varepsilon)\sum_{\{\gamma\}}\sum_{s=1}^{\infty}\frac{\ln\lambda_\gamma\operatorname{Tr}\chi(\gamma^s)}{4\pi|\lambda_\gamma^s - \lambda_\gamma^{-s}|}\int_{-\infty}^{+\infty} h^+(\rho)\lambda_\gamma^{i\rho s}\,d\rho$$

$$+ (1+\varepsilon)\sum_{\{\gamma\}}\sum_{s=1}^{k-1}\frac{1}{16k}\frac{\operatorname{Tr}\chi(\gamma^s)}{\sin\frac{\pi s}{k}}\int_{-\infty}^{+\infty} h^+(\rho)\frac{\cosh\frac{2s-k}{2k}\pi\rho}{\cosh\frac{\pi\rho}{2}}\,d\rho; \tag{1}$$

$$\sum h^-(\rho_k) = \frac{1}{4\pi^2}\nu\mu(\Gamma\setminus G)(1-\varepsilon)\int_{-\infty}^{+\infty} h^-(\rho)\rho\coth\frac{\pi\rho}{2}\,d\rho$$
$$+ (1-\varepsilon)\sum_{\{\gamma\}}\sum_{s=1}^{\infty}\frac{\ln^\lambda\gamma\,\mathrm{Tr}\,\chi(\gamma^s)}{4\pi\,|\lambda_\gamma^s-\lambda_\gamma^{-s}|}\int_{-\infty}^{+\infty} h^-(\rho)\lambda_\gamma^{i\rho s}\,d\rho$$
$$+ (1-\varepsilon)\sum_{\{\gamma\}}\sum_{s=1}^{k-1}\frac{1}{16k}\frac{\mathrm{Tr}\,\chi(\gamma^s)}{\sin\frac{\pi s}{k}}\int_{-\infty}^{+\infty} h^-(\rho)\frac{\sinh\frac{2s-k}{2k}\pi\rho}{\sinh\frac{\pi\rho}{2}} \tag{2}$$

We mention that the sum $\sum h^+(i\rho_l)$ in formula (1) is finite, since $T(g)$ can contain only a finite number of representations of the supplementary series.†

It can be proved further that the number of primitive classes $\{\gamma\}$ of elliptic elements is finite; therefore the contribution of the elliptic elements to the trace formula contains only a finite number of terms.

9. Construction of the Functions $\varphi_n^+(g)$ and $\varphi_n^-(g)$. In **§ 5.7** the formulae for the multiplicity of the representations of the discrete series were obtained on the basis of the following, and so far unproved, proposition: For every natural number n there exist continuous positive definite functions $\varphi_n^-(g)$ and $\varphi_n^+(g)$ (on G) satisfying the following conditions:

1. $T_{\varphi_n^\pm} = \int \varphi_n^\pm(g)T(g)\,dg$ are completely continuous integral operators.

2. For the functions $\varphi_n^+(g)$ we have $h^+(\rho) = h^-(\rho) = 0$, $h_i^- = 0$, $h_j^+ = 0$ for $j \neq n$, $h_n^+ \neq 0$. A similar condition holds for the functions $\varphi_n^-(g)$.

We now describe a construction of such functions under the assumption that $n > 1$. (The excluded cases $n = 0$ and $n = 1$ require a special discussion which we omit.)

For the sake of exactness let us consider the representation $T_n^-(g)$. We recall that it is realized in the space H_n^+ of functions $f(z)$, analytic in the half-plane $\mathrm{Im}\,z > 0$, for which

$$\|f\|^2 = \int_{\mathrm{Im}\,z>0} |f(z)|^2(\mathrm{Im}\,z)^{n-1}\,dx\,dy < +\infty. \tag{1}$$

The representation operator is given by the following formula:

$$T_n^+(g)f(z) = (zg_{12}+g_{22})^{-n-1}f\left(\frac{g_{11}z+g_{21}}{g_{12}z+g_{22}}\right). \tag{2}$$

† Otherwise the set of numbers ρ_l has an accumulation point (because $0 < \rho_l < 1$), and this contradicts the theorem on the discreteness of the spectrum (see p. 26).

We now realize this representation in the space of functions on G itself.

As a preliminary we introduce parameters on G. We specify the matrix $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ by the complex number

$$z = \frac{g_{11}i + g_{21}}{g_{12}i + g_{22}} \tag{3}$$

and the real number

$$\theta = \arg (g_{22} - g_{12}i). \tag{4}$$

Note that $\operatorname{Im} z > 0$.

It is easy to check that under the translation $g \to ga$, $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ the parameters z and θ are transformed according to the following formulae:

$$z \to \frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}, \quad \theta \to \theta - \arg (a_{12}z + a_{22}). \tag{5}$$

Next, we express the invariant measure dg on G in terms of $z = x + iy$ and θ:

$$dg = \frac{1}{2\pi} y^{-2}\, dx\, dy\, d\theta. \tag{6}$$

We substitute for the functions $f(z)$ from the representation space H_n^+ the functions $\varphi(g)$ on G that are defined by the following formula:

$$\varphi(g) = e^{i(n+1)\theta} y^{(n+1)/2} f(z), \tag{7}$$

where $z = x + iy$ and θ is defined by (3) and (4). We have to establish some properties of $\varphi(g)$.

Obviously, $\varphi(g)$ is continuous. Next, on the basis of (6) and (7) we have:

$$\int |\varphi(g)|^2\, dg = \int |f(z)|^2 y^{n-1}\, dx\, dy < \infty. \tag{8}$$

Thus, $\varphi(g)$ of integrable square modulus. Finally, it is easy to check that $\varphi(g)$ is transformed under $f(z) \to T_n^+(g_0) f(z)$ according to the formula

$$T_n^+(g_0)\, \varphi(g) = \varphi(gg_0). \tag{9}$$

So we have obtained a realization of the representation $T_n^+(g)$ in a certain subspace of continuous functions $\varphi(g)$ on G with integrable square modulus. The representation operator in this space is defined by (9).

Let $f_0(z)$ be a vector of dominant weight in H_n^+:

$$f_0(z) = (z + i)^{-n-1}$$

(see **§ 4.5**). We define a function $\varphi_n^+(g)$ by the following formula:

$$\overline{\varphi_n^+(g)} = e^{i(n+1)\theta} y^{(n+1)/2} f_0(z). \tag{10}$$

The function $\varphi_n^+(g)$ is positive definite; this follows from the easily verified equation

$$\varphi_n^+(g) = c(T(g)f_0, f_0).$$

We now show that $\varphi_n^+(g)$ satisfies the estimate

$$|\varphi_n^+(g_0)| < \int_U \tilde{\varphi}(g_0 g)\, dg, \tag{11}$$

where U is a compact neighborhood of the unit matrix, and $\tilde{\varphi}(g)$ is a nonnegative function summable on G.

First of all, we observe that for $n > 1$

$$\int |\varphi_n^+(g)|\, dg = \int y^{(n-3)/2} |z + i|^{-n-1}\, dx\, dy < +\infty, \tag{12}$$

that is, $\varphi_n^+(g)$ is a summable function.

Next, we observe that

$$|\varphi_n^+(ga)| = y^{(n+1)/2} |a_{11} + ia_{12}|^{-n-1} |z + z_a|^{-n-1}, \tag{13}$$

where $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $z_a = \dfrac{a_{21} + ia_{22}}{a_{11} + ia_{12}}$.

Now let U be a compact neighborhood of the unit matrix. From (13) it is clear that there exists a constant $C > 0$, depending on U, such that

$$|\varphi_n^+(g_0)| < C\, |\varphi_n^+(g_0 a)| \tag{14}$$

for every $g_0 \in G$ and every $a \in U$. Consequently $\varphi_n^+(g)$ satisfies the estimate

$$|\varphi_n^+(g_0)| < \int_U \tilde{\varphi}(g_0 g)\, dg,$$

where $\tilde{\varphi}(g) = \dfrac{C}{\text{mes } U} |\varphi_n^+(g)|$; here $\tilde{\varphi}(g)$ is a summable function on G. From this estimate it follows (see **§ 2.2**) that

$$T_{\varphi_n^+} = \int \varphi_n^+(g)\, T(g)\, dg$$

is a completely continuous integral operator. So the validity of condition 1 for the functions $\varphi_n^+(g)$ is established.

Now we show that the functions $\varphi_n^+(g)$ also satisfy condition 2. For by construction, $\overline{\varphi_n^+(g)}$ is contained in an irreducible subspace of

the space of all functions $\varphi(g)$ on G for which

$$\int |\varphi(g)|^2\, dg < \infty.$$

The irreducible representation $T_n^+(g)$ of G acts in this subspace. From this it follows that for every irreducible representation $T_\alpha(g)$ of G, other than $T_n^+(g)$, the operator

$$T_{\varphi_n^+,\alpha} = \int \varphi_n^+(g)\, T_\alpha(g)\, dg$$

is zero. Hence, the trace of $T_{\varphi_n^+,\alpha}$ is zero. So we have shown that for $\varphi_n^+(g)$ we have

$$h^+(\rho) = h^-(\rho) = 0, \qquad h_i^- = 0 \quad \text{and} \quad h_j^+ = 0 \quad \text{for} \quad j \neq n.$$

Now we show that $h_n^+ \neq 0$. For if $h_n^+ = 0$, then on the basis of formula (2) in **§ 5.5**, which expresses $\varphi(e)$ in terms of $h^+(\rho)$, $h^-(\rho)$, h_i and h_j, we have $\varphi_n^+(e) = 0$. However, $\varphi_n^+(e) \neq 0$.

So we have shown that the function $\varphi_n^+(g)$ defined by (10) satisfies the conditions 1 and 2.

The function $\varphi_n^-(g)$ is constructed similarly.

10. The Asymptotic Formula. The formula on pp. 78–79 leads to the following asymptotic formula. Suppose, for the sake of exactness, that $\varepsilon = 1$ so that $T(g)$ does not contain representations of the second principal series. We denote by $N(t)$ the number of ρ_k in the interval $0 \leq \rho_k \leq t$. Then

$$\lim_{t \to +\infty} \frac{N(t)}{t^2} = \frac{1}{4\pi^2}\, \nu\mu(\Gamma \setminus G), \tag{1}$$

where ν is the dimension of the representation $\chi(\kappa)$, $\mu(\Gamma \setminus G)$ is the volume of the factor space $\Gamma \setminus G$. We note that in the absence of elliptic elements this formula enables us to find the genus of Γ, because the genus is uniquely determined by $\mu(\Gamma \setminus G)$.

The idea behind the proof of this formula is as follows. If in the formula on pp. 78–79 we can substitute a function of the following form:

$$\tilde{h}_T(\rho) = \begin{cases} 1, & |\rho| \leq T, \\ 0, & |\rho| > T \end{cases} \tag{2}$$

and then show that the dominant term on the right-hand side is

$$\int_{-\infty}^{\infty} \tilde{h}_T(\rho)\, \rho \tanh \frac{\pi\rho}{2}\, d\rho, \tag{3}$$

then we obtain (1) at once.

However, among the functions for which the formula on pp. 78–79 is true there are no functions of the form (2). So the idea of the subsequent reasoning is to construct a family of functions $h_T(\rho)$ that approximate, in a certain sense, the sequence of functions $\tilde{h}_T(\rho)$.

Suppose then that $h_T(\rho)$ is a sequence of functions with the following properties:

1. $h_T(\rho)$ is an integral function for which

$$|h_T(\rho)| < C \exp \alpha\, |\mathrm{Im}\,(\rho)|,$$

where $\alpha = \min \ln \lambda_\gamma$, the minimum being taken over all primitive hyperbolic elements $\gamma \in \Gamma$.

2. $h_T(\rho)$ is an even function.

3. $h_T(\rho) \geq 0$ on the real axis.

4. There exists an $\varepsilon > 0$ such that $h_T(\rho) = o(|\rho|^{-2-\varepsilon})$ in the domain as $\mathrm{Re}\,\rho \to \infty$.

5. Let $C_T = \max\limits_{0 \leq \rho \leq T} h_T(\rho)$, $c_T = \min\limits_{0 \leq \rho \leq T} h_T(\rho)$. Then

$$\lim_{T \to +\infty} C_T = \lim_{T \to +\infty} c_T = 1.$$

6. $\displaystyle\int_0^\infty h_T(\rho)\,\rho\, d\rho \sim \frac{T^2}{2} \quad \text{as} \quad T \to +\infty.$

7. $\displaystyle\int_T^\infty |h'_T(\rho)|\, \rho^2\, d\rho = O(T^2) \quad \text{as} \quad T \to +\infty.$

Let us show how to derive the asymptotic formula (1) from the existence of such a family of functions $h_T(\rho)$.

First of all, from 1, 2, and 4, we can deduce that $h^+(\rho) \equiv h_T(\rho)$ is the Fourier transform (see the first formula (8) in **§ 5.2**) of a certain function $\varphi_T(g)$ on G. Moreover, this $\varphi_T(g)$ can be chosen so that $T_{\varphi_T} = \int \varphi_T(g)\, T(g)$ is a completely continuous operator and has a trace; the trace formula (1) on pp. 78–79 is then applicable to $\varphi_T(g)$. We examine the terms of this formula separately.

From 1, it follows that the contribution to the trace formula of the hyperbolic elements is zero. Each of the finite number of terms corresponding to the elliptic elements contains under the integral the factor $\dfrac{\cosh \dfrac{2s-k}{2k}\pi\rho}{\cosh \dfrac{\pi\rho}{2}}$, which decreases exponentially as $\rho \to \infty$. Therefore these terms do not contribute to the asymptotic formula. Next we note that the sum $\sum h_T(i\rho_l)$ in the trace formula contains only a finite number of terms (see p. 79), so that it also does not contribute to the asymptotic estimate.

As a result, when we ignore in the formula (1) on p. 78 the terms that do not contribute to the asymptotic estimate, we obtain

$$\sum h_T(\rho_k) \sim \frac{1}{4\pi^2}\, \nu\mu(\Gamma \setminus G) \int_{-\infty}^{+\infty} h_T(\rho)\,\rho \tanh \frac{\pi\rho}{2}\, d\rho.$$

Using 5 and 7 we find that $\sum h_T(\rho_k) \sim N(T)$. On the other hand, from 6 and 7 it follows that $\int_{-\infty}^{+\infty} h_T(\rho)\,\rho \tanh \frac{\pi\rho}{2}\, d\rho \sim T^2$. Hence the required asymptotic formula (1) follows immediately.

Thus, the asymptotic formula follows from the existence of a sequence $h_T(\rho)$ with the properties 1–7.

A construction of a sequence of such functions $h_T(\rho)$ is not given here, but it is noted that the existence of functions $\varphi_T(g)$ on G, for which $h_T(\rho)$ is a Fourier transform and for which the operator T_{φ_T} is completely continuous and has a trace, is closely connected with the Paley–Wiener theorem on G.†

11. The Trace Formula for the Case When $-e$ Does Not Belong to Γ. All the formulae of the preceding subsections were obtained under the assumption that Γ contains the matrix $-e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. However, there is no difficulty in obtaining similar results for the case when $-e$ does not belong to Γ.

Suppose then that Γ is a discrete subgroup of G such that $X = \Gamma \backslash G$ is a compact space and that the subgroup does not contain the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. We consider the representation $T(g)$ of G induced by a finite-dimensional representation $\chi(\gamma)$ of Γ. Let $\varphi(g)$ be a function on G, and $h^+(\rho)$, $h^-(\rho)$, h_n^+, h_n^- its Fourier transforms defined by the formula (8) in **§ 5.2.** Our aim is to compute the multiplicity with which a representation of the discrete series occurs in $T(g)$ and to obtain relations analogous to those in **§ 5.8** for the function $h^+(\rho)$ and $h^-(\rho)$.

Without detailed computations, the contributions elements of various types in Γ make to the trace formula are:

1. *Hyperbolic elements.* First, we modify the definition of a primitive hyperbolic element. A hyperbolic element γ is called primitive if it is not the power of any other element in Γ. (Thus, we do not require here that the eigenvalues of γ are positive, see p. 67.) By analogy with **§ 5.3** we find:

The contribution to the trace formula of the hyperbolic elements is

$$\sum_{\{\gamma\}} \sum_{s=1}^{\infty} \frac{\ln|\lambda_\gamma| \operatorname{Tr} \chi(\gamma^s)}{2\pi\,|\lambda_\gamma^s - \lambda_\gamma^{-s}|} \int_{-\infty}^{+\infty} (h^+(\rho) + h^-(\rho) \operatorname{sign} \lambda_\gamma^s)\, |\lambda_\gamma|^{i\rho}\, d\rho. \quad (1)$$

† An account of this theorem for the case of the group of complex matrices can be found in *Generalized Functions*, vol. 5 [27]; the case of the group of real matrices is treated by Ehrenpreis and Mautner (see [5] in the Bibliography to vol. 5).

Here the notation is the same as in **§ 5.3**; the sum is taken over the set of all primitive classes $\{\gamma\}$ of hyperbolic elements.

2. *Elliptic elements.* The definition of a primitive elliptic element remains unchanged (see p. 70). We note that in our case all elliptic elements are of odd order. For if an element γ in Γ is of even order $2k$, then $\gamma^k = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Consequently, the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ belongs to Γ, which is not the case. By analogy with **§ 5.4** we find:

The contribution to the trace formula of the elliptic elements is

$$-\sum_{\{\gamma\}}\sum_{s=1}^{2k}\frac{1}{2(2k+1)i}\,\frac{\operatorname{Tr}\chi(\gamma^s)}{\sin\dfrac{2\pi s}{2k+1}}\left\{\frac{h_0^+-h_0^-}{2}+\sum_{n=1}^{\infty}\left(h_n^+e^{i2\pi sn/(2k+1)}-h_n^-e^{-i2\pi sn/(2k+1)}\right)\right\}$$

$$+\sum_{\{\gamma\}}\sum_{s=1}^{k}\frac{1}{8(2k+1)}\,\frac{\operatorname{Tr}\chi(\gamma^s)+\operatorname{Tr}\chi(\gamma^{-s})}{\sin\dfrac{2\pi s}{2k+1}}$$

$$\int_{-\infty}^{+\infty}\left(h^+(\rho)\,\frac{\cosh\dfrac{\pi(4s-2k-1)\rho}{2(2k+1)}}{\cosh\dfrac{\pi\rho}{2}}\right.$$

$$\left.-\,h^-(\rho)\,\frac{\sinh\dfrac{\pi(4s-2k-1)\rho}{}}{\sinh\dfrac{\pi\rho}{2}}\right)d\rho. \tag{2}$$

Here the sum is taken over the set of all primitive classes $\{\gamma\}$ of elliptic elements.

3. *The unit element.* The contribution of this element is

$$\nu\mu(\Gamma\setminus G)\,\varphi(e)=\frac{\nu\mu(\Gamma\setminus G)}{\pi^2}\left\{\frac{1}{4}\int_{-\infty}^{+\infty}\rho\tanh\frac{\pi\rho}{2}\,h^+(\rho)\,d\rho\right.$$

$$\left.+\,\frac{1}{4}\int_{-\infty}^{+\infty}\rho\coth\frac{\pi\rho}{2}\,h^-(\rho)\,d\rho+\sum_{n=1}^{\infty}n(h_n^++h_n^-)\right\}. \tag{3}$$

We emphasize that in this case the decomposition of $T(g)$ may contain irreducible representations of all series.

By analogy with **§ 5.7** we find:

The multiplicities N_n^+ and N_n^- with which the representations $T_n^+(g)$ and $T_n^-(g)$ of the discrete series ($n > 0$) occur in $T(g)$ are expressed by the following formulae:

$$N_n^+ = \frac{1}{\pi^2}\,\nu\mu(\Gamma\setminus G)n - \sum_{\{\gamma\}}\sum_{s=1}^{2k}\frac{1}{2(2k+1)i}\,\frac{\operatorname{Tr}\chi(\gamma^s)}{\sin\dfrac{2\pi s}{2k+1}}\,e^{i2\pi sn/(2k+1)}, \tag{4}$$

$$N_n^- = \frac{1}{\pi^2}\,\nu\mu(\Gamma\setminus G)n + \sum_{\{\gamma\}}\sum_{s=1}^{2k}\frac{1}{2(2k+1)i}\,\frac{\operatorname{Tr}\chi(\gamma^s)}{\sin\dfrac{2\pi s}{2k+1}}\,e^{-i2\pi sn/(2k+1)}. \tag{4'}$$

For the functions $h^+(\rho)$ and $h^-(\rho)$ the following relations hold:

$$\sum h^+(\rho_k) + \sum h^+(i\rho_l) = \frac{1}{4\pi^2}\,\nu\mu(\Gamma\setminus G)\int_{-\infty}^{+\infty} h^+(\rho)\,\rho\tanh\frac{\pi\rho}{2}\,d\rho$$

$$+ \sum_{\{\gamma\}}\sum_{s=1}^{\infty}\frac{\ln|\lambda_\gamma|\operatorname{Tr}\chi(\gamma^s)}{2\pi\,|\lambda_\gamma^s - \lambda_\gamma^{-s}|}\int_{-\infty}^{+\infty} h^+(\rho)\,|\lambda_\gamma|^{i\rho s}\,d\rho + \sum_{\{\gamma\}}\sum_{s=1}^{k}\frac{1}{8(2k+1)}$$

$$\frac{\operatorname{Tr}\chi(\gamma^s) + \operatorname{Tr}\chi(\gamma^{-s})}{\sin\dfrac{2\pi s}{2k+1}}\int_{-\infty}^{+\infty} h^+(\rho)\,\frac{\cosh\dfrac{\pi(4s-2k-1)\rho}{2(2k+1)}}{\cosh\dfrac{\pi\rho}{2}}\,d\rho, \tag{5}$$

$$\sum h^-(\rho_k) = \frac{1}{4\pi^2}\,\nu\mu(\Gamma\setminus G)\int_{-\infty}^{+\infty} h^-(\rho)\,\rho\coth\frac{\pi\rho}{2}\,d\rho$$

$$+ \sum_{\{\gamma\}}\sum_{s=1}^{\infty}\frac{\ln|\lambda_\gamma|\operatorname{Tr}\chi(\gamma^s)}{2\pi\,|\lambda_\gamma^s - \lambda^{-s}|}\int_{-\infty}^{+\infty} h^-(\rho)\,|\lambda_\gamma|^{i\rho s}\operatorname{sign}\lambda_\gamma^s\,d\rho$$

$$- \sum_{\{\gamma\}}\sum_{s=1}^{k}\frac{1}{8(2k+1)}\,\frac{\operatorname{Tr}\chi(\gamma^s) + \operatorname{Tr}\chi(\gamma^{-s})}{\sin\dfrac{2\pi s}{2k+1}}$$

$$\int_{-\infty}^{+\infty} h^-(\rho)\,\frac{\sinh\dfrac{\pi(4s-2k-1)\rho}{2(2k+1)}}{\sinh\dfrac{\pi\rho}{2}}\,d\rho. \tag{6}$$

APPENDIX I TO § 5

A Theorem on Continuous Deformations of a Discrete Subgroup

Throughout this appendix Γ denotes a discrete subgroup of the group G of real unimodular matrices of order 2 for which the space $\Gamma \setminus G$ is compact.

In this appendix we try to clarify the degree to which the representation of G generated by $X = \Gamma \setminus G$ determines Γ. To all appearances, although this is so far unproved, the representation determines Γ to within transition to a conjugate subgroup. Here we obtain a somewhat weaker result.

To begin with, we note that if two representations $T_1(g)$ and $T_2(g)$ of G, generated by the spaces $X_1 = \Gamma_1 \setminus G$ and $X_2 = \Gamma_2 \setminus G$, are equivalent and Γ_1 and Γ_2 do not contain elliptic elements, then these groups are isomorphic.

If the representations generated by X_1 and X_2 are equivalent, the indices ρ_i of the representations of the principal series occurring in the decomposition of these representations are identical. But we conclude on the basis of the asymptotic formula proved in **§ 5.10** that the genera of Γ_1 and Γ_2 are the same. We then apply the standard result that if two subgroups Γ_1 and Γ_2 have the same genus they are isomorphic.

Now we introduce the notion of a *continuous deformation* of a subgroup Γ. We assume that every t in the interval $0 \leq t \leq 1$ is associated with a discrete subgroup $\Gamma_t \subset G$, isomorphic to Γ, and $\Gamma_0 = \Gamma$. By $\gamma(t)$ we denote the image of the element $\gamma \in \Gamma$ under the isomorphism $\Gamma \to \Gamma_t$. If all the functions $\gamma(t)$ are continuous, then the family of groups Γ_t is called a continuous deformation of the group $\Gamma = \Gamma_0$.

Every group Γ has a continuous deformation of the following form. Let $g(t)$, $0 \leq t \leq 1$, be a continuous curve starting from the unit element ($g(0) = e$) in G. We set $\Gamma_t = g^{-1}(t)\,\Gamma g(t)$. Clearly the family of groups Γ_t forms a continuous deformation of Γ; such a deformation is called *trivial.*

It is known that for discrete subgroups (with a compact fundamental domain) of an arbitrary semisimple Lie group other than the group of real matrices of order 2, every continuous deformation is trivial. This remarkable result is due to A. Weil.

In this appendix we prove the following theorem.

THEOREM. *Let Γ_t be a continuous deformation of Γ. If the representations of G generated by the spaces $X_t = \Gamma_t \setminus G$ are equivalent, then the deformation Γ_t is trivial.*

We shall use the following fact.

(*) *The traces of the matrices of the discrete subgroup form a discrete set on the real line.*†

On the basis of this fact we shall now prove two lemmas.

LEMMA 1. *If the representations $T_1(g)$ and $T_2(g)$ generated by the spaces $X_1 = \Gamma_1 \setminus G$ and $X_2 = \Gamma_2 \setminus G$ are equivalent, then for every matrix $\gamma_1 \in \Gamma_1$ we can find a matrix $\gamma_2 \in \Gamma_2$ and conjugate to γ_1.*

Proof. If $T_1(g)$ and $T_2(g)$ are equivalent, then these two representations have the same trace. Consequently, for every finite continuous function $\varphi(g)$ we have

$$\int_{\Gamma_1\setminus G} \sum_{\gamma\in\Gamma_1} \varphi(g^{-1}\gamma g)\, dg = \int_{\Gamma_2\setminus G} \sum_{\gamma\in\Gamma} \varphi(g^{-1}\gamma g)\, dg \tag{1}$$

(see **§ 2.4** (3)). Now take $\gamma_1 \in \Gamma_1$.

Suppose that Γ_2 contains no element conjugate to γ_1. Then, by (*), for every function $\varphi(g)$ that is concentrated in a sufficiently small neighborhood of γ_1 the right-hand side of (1) is equal to zero. But this is impossible, because for such functions $\varphi(g)$ the left-hand side of (1) is different from zero.

LEMMA 2. *Let Γ_t be a continuous deformation of G such that the representations generated by the homogeneous spaces $X_t = \Gamma_t \setminus G$ are all equivalent.*

We denote by $\gamma(t)$ the image of the element $\gamma = \gamma(0) \in \Gamma$ under the isomorphism $\Gamma \to \Gamma_t$. Then the trace of the matrix $\gamma(t)$ does not depend on t.

Proof. Clearly it is sufficient to prove the statement of the lemma for sufficiently small values of t. But for small t the assertion follows immediately from Lemma 1 and from (*).

Now we pass on to the proof of the theorem. We denote by $\mathfrak{A}_t$ the algebra of matrices of the form $\sum_k \lambda_k\gamma_k(t)$, where the λ_k are real numbers and the $\gamma_k(t)$ elements of Γ_t.

It is not hard to verify that $\mathfrak{A}_t$ coincides with the algebra $\mathfrak{A}$ of all matrices of order 2.

We set up an isomorphic map $\mathfrak{A}_0 \to \mathfrak{A}_t$ of $\mathfrak{A}_0$ onto $\mathfrak{A}_t$. Let $a \in \mathfrak{A}_0$. We represent a in the form

$$a = \sum_k \lambda_k\gamma_k, \tag{2}$$

where $\gamma_k \in \Gamma$, and associate with it the element $a_t \in \mathfrak{A}_t$ of the form

$$a_t = \sum_k \lambda_k\gamma_k(t).$$

---

† If the sequence $\{\mathrm{Tr}\,\gamma_k\}$ has a limit, then for a suitable choice of $g_k \in G$ the sequence $g_k^{-1}\gamma_k g_k$ also has a limit. We write g_k in the form $g_k = \gamma'_k\gamma'_k$, where $\gamma'_k \in \Gamma$, $g'_k \in F$. Since the fundamental domain F is compact, we may assume that $\{g'_k\}$ is a convergent sequence. But then $\{(\gamma'_k)^{-1}\gamma_k\gamma'_k\}$ is also convergent, and this contradicts the fact that Γ is discrete.

We have to verify that the correspondence $a \to a_t$ does not depend on the expression (2) for a. In other words, we have to show that if $\sum_k \lambda_k \gamma_k = 0$, then also $\sum_k \lambda_k \gamma_k(t) = 0$.

Suppose that $\sum_k \lambda_k \gamma_k = 0$. Then $\sum_k \lambda_k \gamma_k \gamma = 0$ for every $\gamma \in \Gamma$, and so $\operatorname{Tr}\left(\sum_k \lambda_k \gamma_k \gamma\right) = 0$. But by Lemma 2,

$$\operatorname{Tr}\left(\sum_k \lambda_k \gamma_k \gamma\right) = \operatorname{Tr}\left(\sum_k \lambda_k \gamma_k(t) \gamma(t)\right).$$

Consequently,

$$\operatorname{Tr}\left(\sum_k \lambda_k \gamma_k(t) \gamma(t)\right) = 0 \tag{3}$$

for every $\gamma(t) \in \Gamma_t$. Since the algebra spanned by the matrices $\gamma(t)$ coincides with $\mathfrak{A}$, it follows from (3) that

$$\operatorname{Tr}\left(\left(\sum_k \lambda_k \gamma_k(t)\right) a\right) = 0$$

for every matrix a. Consequently,

$$\sum_k \lambda_k \gamma_k(t) = 0.$$

So we have defined a map of $\mathfrak{A}_0$ onto $\mathfrak{A}_t$. It is easy to see that this is an isomorphism.

It is well known that every automorphism of the complete matrix algebra is an inner automorphism. Hence there exists a matrix $g = g(t)$ such that

$$\sum_k \lambda_k \gamma_k(t) = g^{-1}(t)\left(\sum_k \lambda_k \gamma_k\right) g(t) \tag{4}$$

for arbitrary real numbers λ_k and arbitrary elements $\gamma_k \in \Gamma$.

From the fact that the left-hand side of (4) is a continuous function of t it follows that $g(t)$ may also be chosen as a continuous function of t.

In particular, for every matrix $\gamma \in \Gamma$ we find on the basis of (4) that

$$\gamma(t) = g^{-1}(t) \gamma g(t),$$

that is, Γ_t is the trivial deformation. The proof of the theorem is now complete.

APPENDIX II TO §5

The Trace Formula for the Group of Complex Unimodular Matices of Order 2

Here we derive the trace formula for the group G of complex unimodular matrices of order 2. The complex case turns out to be considerably simpler than the real one, because it is easier to construct irreducible representations for the group of complex matrices than for the group of real matrices. The reader will see in **2.** that the trace formula for the group of complex matrices has a much simpler structure than in the real case.

Since the results for the complex case are obtained by the same methods as those used in the real case, the details of the proofs are omitted.

1. Irreducible Unitary Representations of G**.** The group G of complex unimodular matrices of order 2 has two series of irreducible unitary representations—the principal and the supplementary series.

The principal series of representations is realized in the space H of functions $f(z)$ of a complex variable z of integrable square modulus

$$\frac{i}{2}\int |f(z)|^2\, dz\, d\bar{z} < +\infty.$$

The representation consists in associating with every complex matrix

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

an operator $T_{\rho,m}(g)$ in H of the following form:

$$T_{\rho,m}(g)f(z) = |\beta z + \delta|^{m+i\rho-2}\,(\beta z + \delta)^{-m} f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{1}$$

Thus, the representation is given by a pair of numbers: a real number ρ and an integer m. It can be shown that two representations $T_{\rho,m}(g)$ and $T_{\rho',m'}(g)$ are equivalent if and only if either $\rho = \rho'$, $m = m'$ or $\rho = -\rho'$, $m = -m'$.

The character of $T_{\rho,m}(g)$ is given by the following formula:

$$\sigma_{\rho,m}(g) = \frac{|\lambda_g|^{i\rho+m}\,\lambda_g^{-m} + |\lambda_g|^{-i\rho-m}\lambda_g^{m}}{|\lambda_g - \lambda_g^{-1}|^2} \tag{2}$$

where λ_g and λ_g^{-1} are the eigenvalues of g.

If in (1) we allow ρ to be an arbitrary complex number, then we obtain, in general, nonunitary representations of G. There is

always a reasonable way of giving a space of functions $f(z)$ in which this representation acts. The character of $T_{\rho,m}(g)$ in this general case is also given by (2).

The supplementary series of irreducible unitary representations is obtained for $m = 0$, $\rho = is$, where $-2 < s < 2$, $s \neq 0$. The Hilbert space in which the operators $T_{\rho,0}(g)$ of the supplementary series act consists of the functions $f(z)$ for which

$$\|f\|^2 = \left(\frac{i}{2}\right)^2 \int |z_1 - z_2|^{s-2} f(z_1)\overline{f(z_2)}\, dz_1\, d\overline{z_1}\, dz_2\, d\overline{z_2} < +\infty.$$

The reader will find details on irreducible representations of G in Chapter II, and also in [28].

2. The Trace Formula for G. Let Γ be a discrete subgroup of G such that $\Gamma \setminus G$ is a compact space; $\chi(\gamma)$ a finite-dimensional representation of γ; $T(g)$ the representation of G induced by $\chi(\gamma)$. Since $\Gamma \setminus G$ is a compact space, the representation $T(g)$ splits into a direct sum of irreducible unitary representations. Information on what irreducible representations occur in this decomposition is contained in the trace formula of **§ 2**:

$$\sum \int \varphi(g)\sigma_{\rho_k,m_k}(g)\, dg = \sum_{\gamma\in\Gamma} \operatorname{Tr} \chi(\gamma)\mu(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma\setminus G} \varphi(g^{-1}\gamma g)\, dg, \quad (1)$$

where the summation on the left is taken over all representations that occur in $T(g)$.†

This formula is true for any function $\varphi(g)$ on G for which $T_\varphi = \int \varphi(g)T(g)\, dg$ is a completely continuous operator having a trace, in particular, for every smooth finite positive definite function.

Our aim is to pass in the trace formula (1) from the function $\varphi(g)$ to its Fourier transform

$$h(\rho, m) = \int \varphi(g)\sigma_{\rho,m}(g)\, dg.$$

By the formulae for the characters $\sigma_{\rho,m}(g)$ under **1** this function $h(\rho, m)$ is given by the following explicit formula:

$$h(\rho, m) = \int \varphi(g) \frac{|\lambda_g|^{i\rho+m}\lambda_g^{-m} + |\lambda_g|^{-i\rho-m}\lambda_g^{m}}{|\lambda_g - \lambda_g^{-1}|^2}\, dg. \quad (2)$$

The left-hand side of (1) is transformed to $h(\rho, m)$ in an obvious way. So our problem is to express in terms of $h(\rho, m)$ the integrals

$$\int_{G_\gamma\setminus G} \varphi(g^{-1}\gamma g)\, dg$$

over the class of elements conjugate to the matrix $\gamma \in \Gamma$.

† We recall that by Γ_γ and G_γ in (1) we denote the centralizers of the element $\gamma \in \Gamma$ in Γ and G, respectively.

Note that owing to the compactness of $\Gamma \setminus G$, the group Γ has no parabolic elements. Hence every element $\gamma \in \Gamma$, other than $\pm e$, is conjugate to a diagonal matrix

$$\delta = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

We introduce the function

$$I(\delta) = \int\limits_{G_\delta \setminus G} \varphi(g^{-1}\,\delta g)\, dg$$

and try to express it in terms of $h(\rho, m)$. For this purpose we use an integral relation between the functions $\varphi(g)$ and $I(\delta)$, which we give without proof†

$$\frac{1}{2}\cdot\frac{i}{2}\int I(\delta)\,\omega(\delta)\,\frac{d\lambda\, d\bar{\lambda}}{|\lambda|^2} = \int \varphi(g)\,\omega(g)\,|\lambda_g - \lambda_g^{-1}|^{-4}\, dg,$$

where $\omega(g)$ is an arbitrary function on G that is constant on classes of conjugate elements.

When in this relation we set

$$\begin{aligned}\omega(g) &= |\lambda_g - \lambda_g^{-1}|^4\, \sigma_{\rho,m}(g) \\ &= |\lambda_g - \lambda_g^{-1}|^2\,(|\lambda_g|^{i\rho+m}\lambda_g^{-m} + |\lambda_g|^{-i\rho-m}\lambda_g^{m}),\end{aligned}$$

we find that

$$\frac{1}{2}\cdot\frac{i}{2}\int I(\delta)\,|\lambda - \lambda^{-1}|^2\,(|\lambda|^{i\rho+m}\,\lambda^{-m} + |\lambda|^{-i\rho-m}\,\lambda^{m})\,\frac{d\lambda\, d\bar{\lambda}}{|\lambda|^2} = h(\rho, m).$$

Bearing in mind that $I(\delta) = I(\delta^{-1})$, we can rewrite this formula in the form

$$\frac{i}{2}\int I(\delta)\,|\lambda - \lambda^{-1}|^2\,|\lambda|^{i\rho+m}\lambda^{-m}\,\frac{d\lambda\, d\bar{\lambda}}{|\lambda|^2} = h(\rho, m).$$

So we see that $h(\rho, m)$ is the Mellin transform of the function $I(\lambda)\,|\lambda - \lambda^{-1}|^2$. Consequently, by the formula for the inverse Mellin transform, we have

$$I(\delta) = \frac{(2\pi)^{-2}}{|\lambda - \lambda^{-1}|^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{+\infty} h(\rho, m)\,|\lambda|^{-i\rho-m}\,\lambda^{m}\, d\rho.$$

So we have expressed the integral over every proper class of conjugate elements in terms of $h(\rho, m)$. On the basis of this formula we have: for $\gamma \neq \pm e$

$$\int\limits_{G_\gamma \setminus G} \varphi(g^{-1}\gamma g)\, dg = \frac{(2\pi)^{-2}}{|\lambda_\gamma - \lambda_\gamma^{-1}|^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{+\infty} h(\rho, m)\,|\lambda_\gamma|^{-i\rho-m}\,\lambda_\gamma^{m}\, d\rho,$$

where λ_γ and λ_γ^{-1} are the eigenvalues of γ.

† A derivation of this relation can be found, for example, in [28].

Now we examine the exceptional case: $\gamma = e$ and $\gamma = -e$. In this case the class of conjugate elements consists of a single element, and the integral over this class degenerates into $\varphi(e)$ and $\varphi(-e)$, respectively. So we have to express $\varphi(e)$ and $\varphi(-e)$ in terms of $h(\rho, m)$. Without proof we give the final formulae

$$\varphi(e) = \frac{1}{32\pi^4} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{+\infty} (m^2 + \rho^2) h(\rho, m)\, d\rho,$$

$$\varphi(-e) = \frac{1}{32\pi^4} \sum_{m=-\infty}^{\infty} (-1)^m \int_{-\infty}^{\infty} (m^2 + \rho^2) h(\rho, m)\, d\rho.$$

These formulae immediately follow from results of Chapter 2, **§ 6**; see also [28].

As a result, after going over from the functions $\varphi(g)$ to $h(\rho, m)$ the trace formula assumes the form

$$\sum h(\rho_k, m_k) = \frac{1}{32\pi^4} \nu\mu(\Gamma \setminus G)$$
$$\sum_{m=-\infty}^{\infty} [1 + (-1)^m \varepsilon] \int_{-\infty}^{\infty} (m^2 + \rho^2) h(\rho, m)\, d\rho$$
$$+ \sum_{\gamma\in\Gamma} \frac{\operatorname{Tr} \chi(\gamma) \mu(\Gamma_\gamma \setminus G_\gamma)}{4\pi^2 |\lambda_\gamma - \lambda_\gamma^{-1}|^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{+\infty} h(\rho, m) |\lambda_\gamma|^{-i\rho - m} \lambda^m\, d\rho \quad (3)$$

where ν is the dimension of the representation $\chi(\gamma)$; $\varepsilon = 1$ when $\chi(-e) = E$, $\varepsilon = -1$ when $\chi(-e) = -E$, E is the unit operator.

The summation on the left is over all the representations of the principal and the supplementary series that are contained in $T(g)$.

The formula can be split into relations for $h(\rho, m)$ for fixed values of m:

$$\sum_k h(\rho_k, 0) = \frac{1}{32\pi^4} \nu\mu(\Gamma \setminus G)(1 + \varepsilon) \int_{-\infty}^{+\infty} \rho^2 h(\rho, 0)\, d\rho$$
$$+ \sum_{\gamma\in\Gamma} \frac{\operatorname{Tr} \chi(\gamma) \mu(\Gamma_\gamma \setminus G_\gamma)}{4\pi^2 |\lambda_\gamma - \lambda_\gamma^{-1}|^2} \int_{-\infty}^{+\infty} h(\rho, 0) |\lambda_\gamma|^{-i\rho}\, d\rho; \quad (4)$$

$$\sum_k h(\rho_{m,k}, m) = \frac{1}{16\pi^4} \nu\mu(\Gamma \setminus G)[1 + (-1)^m \varepsilon]$$
$$\int_{-\infty}^{+\infty} (m^2 + \rho^2) h(\rho, m)\, d\rho + \sum_{\gamma\in\Gamma} \frac{2 \operatorname{Tr} \chi(\gamma) \mu(\Gamma_\gamma \setminus G_\gamma)}{4\pi^2 |\lambda_\gamma - \lambda_\gamma^{-1}|^2}$$
$$\int_{-\infty}^{+\infty} h(\rho, m) |\lambda_\gamma|^{-i\rho - m} \lambda_\gamma^m\, d\rho. \quad (5)$$

The representations of the supplementary series are covered only by the first of these formulae. The derivation of the formulae from (3) is left to the reader.

3. The Asymptotic Formula. From (4) and (5) of **2** it is easy to derive asymptotic formulae for the distribution of the numbers ρ_k. We give the formulae here without proof. We denote by $N_m(\rho)$ the set of numbers $\rho_{m,k} > 0$ that do not exceed ρ. Then the following asymptotic formula holds as $\rho \to \infty$:

$$N_m(\rho) \sim \frac{\nu\mu(\Gamma \setminus G)}{12\pi^2} \rho^3.$$

§ 6. INVESTIGATION OF THE SPECTRUM OF A REPRESENTATION GENERATED BY A NON-COMPACT SPACE $X = \Gamma \setminus G$ (SEPARATION OF THE DISCRETE PART OF THE SPECTRUM)

In this section we continue the investigation of the representations of the group G of real matrices of order 2 generated by the spaces $X = \Gamma \setminus G$, where Γ is a discrete subgroup of G. As before, we assume that Γ contains the matrix $-e$.

Earlier we investigated in detail the case of a compact space X. We proved that the spectrum of the representation generated by X is discrete.

Here we assume that X is not a compact space, but has finite volume. The main task, as before, consists in studying the spectrum of the representation $T(g)$ generated by X, in other words, to decompose this representation into irreducible ones. The solution of this problem will be given by the method of horospheres.

The method of horospheres enables us to decompose the representation space into two invariant subspaces having much simpler spectral structures. The first of these subspaces to be studied has a countable discrete spectrum.

It can be shown that the discrete spectrum of the second subspace has only a finite number of points, and that the spectrum of its remaining part is of the multiplicity of the continuum. This multiplicity of the continuous spectrum is equal to the minimal number of parabolic vertices of a fundamental domain of Γ. The proof is based on the perturbation theory of differential operators. To avoid overloading the book with special problems in the theory of differential operators, we give an account of this proof in another place.

1. Horospheres in a Homogeneous Space. Horospherical subgroups of G are subgroups Z of matrices of the form

$$z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

and all subgroups conjugate to Z.

Horospheres in a homogeneous space $X = \Gamma \setminus G$ are orbits of horospherical subgroups, that is, curves of the form

$$x_z = xgzg^{-1},$$

where $x \in X$ and $g \in G$ are fixed, and z ranges over Z.

We note that the majority of horospheres on $\Gamma \setminus G$ turn out to be noncompact and even nonclosed sets. Only those horospheres that are compact in $\Gamma \setminus G$ play an important role for us.

Let us state a compactness condition for horospheres. Since a horosphere is translated into a horosphere and a compact horosphere is translated into a compact horosphere, it is sufficient to restrict our attention to horospheres passing through a fixed point x_0:

$$x_z = x_0 gzg^{-1}.$$

For x_0 we choose the point corresponding to the unit class in $\Gamma \setminus G$.

To begin with, let $g = e$. Then a horosphere has the form

$$x_z = x_0 z. \tag{1}$$

We show that *the horosphere* (1) *is compact if and only if the subgroup* $\Delta = \Gamma \cap Z$ *is not trivial.*

Proof. We consider a continuous map

$$z \to x_0 z \tag{2}$$

of Z onto our horosphere C. Clearly the inverse images of the points of C under this map are cosets of $\Delta = \Gamma \cap Z$ in Z.

In this way the map (2) induces a continuous one-to-one map

$$Z/\Delta \to C$$

of the factor group Z/Δ onto C.

Let us assume that $\Delta = \Gamma \cap Z$ is not trivial, and show that then C is compact. The horospherical subgroup Z is isomorphic to the additive group of real numbers; from this it follows immediately that its factor group with respect to Δ is compact, if the latter is nontrivial. But then C, as a continuous image of the compact set Z/Δ, is also compact.

We now assume, conversely, that C is compact. Then the factor space Z/Δ is compact, and hence Δ is not trivial. This follows immediately from a general theorem, which we quote here without proof (see Pontryagin [57], Chapter 3, Theorem 20).

Let C be a locally compact homogeneous space in which a locally compact group Z having a countable open covering by compact sets acts. Then C is isomorphic to the space Z/Δ of cosets of Z with respect to the stability group Δ of one of the points of C.

So we have established that for compactness of the horosphere

$$x_z = x_0 z$$

it is necessary and sufficient that the subgroup $\Delta = \Gamma \cap Z$ is not trivial.

In a similar way it is easy to check that for compactness of the horosphere

$$x_z = x_0 g z g^{-1}$$

it is necessary and sufficient that the subgroup

$$\Delta_g = \Gamma \cap g Z g^{-1}$$

is not trivial.

We say that two compact horospheres *belong to a family* if each arises from the other by a translation. The number of such families is an important characteristic of the homogeneous space $X = \Gamma \setminus G$. It is not hard to verify that this number is equal to the minimal number of parabolic vertices of a fundamental domain relative to Γ. We shall derive from the last results of this subsection that this number is finite, provided the volume of X is finite.

2. Statement of the Main Theorem. Let Γ be a discrete subgroup of G such that the volume of $\Gamma \setminus G$ is finite.

We consider the representation of G generated by the homogeneous space $X = \Gamma \setminus G$. We recall that this representation acts in the space H of functions $f(x)$, $x \in X$, of integrable square modulus:

$$\int |f(x)|^2 \, dx < \infty.$$

The representation operator is given by the formula

$$T(g) f(x) = f(xg).$$

Our task is to investigate the spectrum of this representation. In this subsection we separate the discrete part of the spectrum. We shall define an invariant subspace H^0 of H whose spectrum is discrete.

First we state the precise result of this subsection.

We consider the collection H^0 of functions from H whose integrals over any compact horosphere are equal to zero. It is not hard to check that H^0 is a closed subspace of H and that it is invariant.

It stands to reason that the condition for the integral over one compact horosphere to be equal to zero by no means singles out a closed subspace. However, the condition that the integrals over a given compact horosphere and all

horospheres sufficiently near to it are equal to zero in fact defines a closed subspace.

Proof. We consider the set of horospheres that are sufficiently close to a given compact horosphere l. Since these horospheres do not intersect (see below in **§ 6.3**), the domain $K \subset X$ they fill is the topological product

$$K = T \times C$$

of a compactum T and C. Let H_K be the space of functions $f(t, c)$ from H that are concentrated on K. The condition that the integrals of these functions over the horospheres close to l are zero can be written in the form

$$\int_C f(t, c)\, dc = 0 \quad \text{for every} \quad t \in T$$

Obviously this condition singles out a closed subspace H'_K in H_K (this can be seen, for example, by going over from the functions $f(t, c)$ to their Fourier transforms with respect to c). But then this condition singles out a closed subspace of the whole space H, namely the subspace $H'_K \dotplus H''_K$, where H''_K is the orthogonal complement to H_K.

THEOREM. *The space H^0 splits into the direct sum of not more than countably many invariant irreducible subspaces.* In other words, the spectrum of $T(g)$ in H^0 is discrete.

Leaning on results of **§ 2** we reduce this theorem to the proof of another proposition.

We consider finite functions $\varphi(g)$ on G of the form

$$\varphi(g) = \psi(g) * \overline{\psi(g^{-1})},$$

where $\psi(g)$ is a finite infinitely differentiable function on G that differs from 0 only in a sufficiently small neighborhood of the unit element.

Our aim is to prove that

$$T_\varphi = \int \varphi(g)\, T(g)\, dg$$

are completely continuous operators in H^0. From this it follows immediately by the Lemma in **§ 2.3** that H^0 splits into the direct sum of not more than countably many invariant irreducible subspaces.

Since T_φ is a self-adjoint positive definite operator, to prove that it is completely continuous it is sufficient to show that its trace is finite.

So the main theorem reduces to the proof of the following proposition. *The operator T_φ, where $\varphi(g)$ is a function of the form $\varphi(g) = \psi(g) * \psi(g^{-1})$, concentrated in a sufficiently small neighborhood of the unit element of G, has a finite trace on H^0.*

3. Cylindrical Sets. For the proof of the main theorem we split the space $X = \Gamma \setminus G$ into *cylindrical* subsets whose structure is in a certain sense simpler than that of X.

We call an open subset X_i of X a *cylindrical set* if it can be covered by pairwise disjoint compact horospheres belonging to one and the same family.

In other words, X_i is the set of all elements $x \in X$ of the form

$$x = x_0 g_0 z g_0^{-1} v,$$

where x_0 is a fixed point in X, g_0 a fixed element of G such that the subgroup $\Gamma \cap g_0 Z g_0^{-1}$ is nontrivial (compactness condition for the horospheres); z ranges over Z and v over a certain set V in G. Here the element v—the index of the horosphere—is uniquely determined by x.

The object of this subsection is to prove the following proposition. *The space X can be represented as a union of a finite number of pairwise disjoint sets*

$$X = X_0 + X_1 + \cdots + X_p,$$

where X_0 is compact, and $X_1, \ldots, X_p$ are cylindrical sets.

The proof of this proposition is based on a result obtained in **§ 1.4**, and now restated.

In **§ 1.4** we showed that on the Lobachevskii plane there exists a fundamental domain F relative to Γ that is the union of pairwise disjoint subsets

$$F = F_0 + \sum_{k=1}^{p} F(b_k), \tag{1}$$

where F_0 is compact, b_k are the vertices of F at infinity, and $F(b_k)$ are the triangles bounded by two geodesic lines l_k, l_k' starting from b_k and a horocycle ω_k, passing through b_k, $k = 1, \ldots, p$. Here the sides l_k and l_k' of $F(b_k)$ are equivalent, that is, are obtained from each other by a transformation $\gamma \in \Gamma$. Moreover, every point inside the horocycle ω_k can be carried into a point of $F(b_k)$ by a transformation $\gamma \in \Gamma$ that leaves b_k in its place.

With the decomposition (1) we associate a decomposition of $X = \Gamma \setminus G$ into disjoint subsets.

For this purpose we observe that the Lobachevskii plane is the homogeneous space G/U of cosets of the group G of real unimodular matrices of order 2 by the subgroup U of orthogonal matrices. Consequently, the fundamental domain F on the Lobachevskii plane relative to Γ can be identified in a natural way with the space of double cosets $\Gamma \setminus G/U$. We consider the natural map

$$X = \Gamma \setminus G \to F = \Gamma \setminus G/U. \tag{2}$$

We denote by X_0 and X_k the complete inverse images of the sets F_0 and $F(b_k)$, $k = 1, \ldots, p$, under this map. Then we obtain the decomposition

$$X = X_0 + X_1 + \cdots + X_p, \tag{3}$$

of X into pairwise disjoint subsets.

It is clear that X_0 is a compact set (because its image F_0 and the kernel of the map U are compact sets). We have to show that the remaining sets X_k in this decomposition are cylindrical.

We examine one of these sets X_k and its image $F(b_k)$ under the map (2). First we describe these sets in matrix form.

Without loss of generality we may assume that $b_k = \infty$. Then the subgroup of parabolic elements leaving b_k fixed coincides with group Z of matrices of the form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. Since there exist parabolic transformations $\gamma \in \Gamma$ leaving the point b_k fixed, the subgroup $\Delta = \Gamma \cap Z$ is nontrivial.

It is easy to verify that the set $F(b_k)$ consists of all points of the form

$$(za)z_0,$$

where $z_0 = i$ is a point of the Lobachevskii plane having a subgroup of U as its stability group, a ranges over the set of diagonal matrices of the form

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \qquad 0 < \alpha < N, \tag{4}$$

and z ranges over a fundamental domain Z_Γ of Z relative to $\Delta = \Gamma \cap Z$. This domain Z_Γ is compact, because Δ is not trivial.

Clearly the complete inverse image X_k of $F(b_k)$ consists of all points of the form

$$x = x_0 zau, \tag{5}$$

where x_0 is a fixed point in $X = \Gamma \setminus G$ corresponding to the unit class, and z, a, and u range over the sets of matrices described above.

Let us show that the X_k are cylindrical sets. First of all we note that for fixed a and u the set of points (5) forms a compact horosphere.

Thus, through every point of X_k there passes a compact horosphere from the given family. It remains to verify that distinct pairs a, u correspond to horospheres without common points.

Suppose that the horospheres $x_z = x_0 z a_1 u_1$ and $x_z = x_0 z a_2 u_2$ have points in common. Then there exist elements $z_1, z_2 \in Z_\Gamma$ and $\gamma \in \Gamma$ such that

$$\gamma z_1 a_1 u_1 = z_2 a_2 u_2. \tag{6}$$

We consider the map $G \to G/U$ of G onto the Lobachevskii plane. Under this map the elements $z_1 a_1 u_1$ and $z_2 a_2 u_2$ go over into

points in $F(b_k)$. The equation (6) means that these points can be carried one into the other by a certain element $\gamma \in \Gamma$. But since $F(b_k)$ belongs to a fundamental domain relative to Γ, this is possible only when these points coincide and $\gamma = 1$.

Thus, from (5) it follows that $\gamma = 1$, and therefore

$$z_1 a_1 u_1 = z_2 a_2 u_2. \tag{7}$$

Since every matrix $g \in G$ may be decomposed in a *unique* way into a product of the form $g = zau$, we see from (7) that $a_1 = a_2$ and $u_1 = u_2$. Consequently the horospheres $x_2 = x_0 z a_1 u_1$ and $x_z = x_0 z a_2 u_2$ coincide.

So we have shown that the sets $X_k, k = 1, \ldots, p$, in the decomposition (3) of the space $X = \Gamma \setminus G$ are cylindrical.

4. Reduction of the Main Theorem. In **§ 6.2** the main theorem of this section was reduced to the following theorem.

Every operator of the form

$$T_\varphi = \int \varphi(g) T(g)\, dg,$$

where $\varphi(g)$ is an infinitely differentiable function of the form

$$\varphi(g) = \psi(g) * \psi(g^{-1}),$$

concentrated in a sufficiently small neighborhood of the unit element of G, has finite trace on H^0

We now make a further reduction of the main theorem.

For this purpose we split the homogeneous space X into the sum of disjoint subsets

$$X = X_0 + X_1 + \cdots + X_p,$$

where X_0 is compact and each of the sets $X_1, \ldots, X_p$ is cylindrical, that is, splits into pairwise disjoint horospheres of one and the same family. The possibility of such a decomposition was established in the preceding subsection.

We denote by H_k the subspace of functions of integrable square modulus on X and equal to zero outside X_k, and by P_k the projection operator of H onto this subset, $k = 0, 1, \ldots, p$. Furthermore we denote by H_k^0 the subspace of functions of H_k whose integrals over the horospheres of the family occurring in X_k are equal to zero.

Obviously we have the following inclusion:

$$H^0 \subset H_0 + H_1^0 + \cdots + H_p^0. \tag{1}$$

Hence it follows that the trace of the positive definite self-adjoint operator T_φ on the space H^0 is not greater than the sum of the traces of the operators $P_k T_\varphi P_k$ on the spaces $H_0, H_1^0, \ldots, H_p^0$.

So the proof of the main theorem reduces to that of the following proposition.

The trace of the operator $P_0 T_\varphi P_0$ on H_0 and the trace of the operator $P_k T_\varphi P_k$ on the spaces H_k^0, $k = 1, \ldots, p$, are finite.

We begin by showing that the trace of $P_0 T_\varphi P_0$ in H_0 is finite (note that X_0 is compact).

For this purpose we recall that T_φ is a positive definite integral operator of the form

$$T_\varphi f(g_1) = \int_F K(g_1, g_2) f(g_2)\, dg_2 \tag{2}$$

with the kernel

$$K(g_1, g_2) = \sum_{\gamma \in \Gamma} \varphi(g_1^{-1} \gamma g_2), \tag{3}$$

where F is a fundamental domain in G relative to the transformation $g \to \gamma g$, $\gamma \in \Gamma$. Also $K(g_1, g_2)$ is a continuous function of g_1 and g_2.

Let F_0 be the inverse image of X_0 in F. The map $F_0 \to X_0$ is one-to-one and bicontinuous, hence F_0 is also a compact set.

It is easy to see that $P_0 T_\varphi P_0$ is an integral operator on F_0 of the form

$$\int_{F_0} K(g_1, g_2) f(g_2)\, dg_2.$$

Since the kernel $K(g_1, g_2)$ is continuous, it is bounded on F_0. Therefore the trace of $P_0 T_\varphi P_0$ on the whole space H_0 is finite, and equal to

$$\int_{F_0} K(g, g)\, dg.$$

It now remains to show that the trace of $P_k T_\varphi P_k$, $k = 1, \ldots, p$, is finite. We pass on to the proof of this assertion.

5. Proof That the Trace of $P_k T_\varphi P_k$ in H_k^0 is Finite. Let F be a fundamental domain of G relative to Γ, and F_k the inverse image of the cylindrical set X_k in F. Without loss of generality we may assume that X_k splits into horospheres of the form

$$x = x_0 z g. \tag{1}$$

As we have shown in **§ 6.3**, F_k then consists of all possible elements of the form

$$zau,$$

where z ranges over a fundamental domain of Z relative to

$$\Delta = \Gamma \cap Z,$$

a ranges over the set of diagonal matrices

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \qquad 0 < \alpha < N, \tag{2}$$

and u over the set of orthogonal matrices.

Obviously, $P_k T_\varphi P_k$ can be regarded in H_k as an integral operator on F_k of the form

$$\int_{F_k} K(g_1, g_2) f(g_2)\, dg_2, \tag{3}$$

where

$$K(g_1, g_2) = \sum_{\gamma \in \Gamma} \varphi(g_1^{-1} \gamma g_2), \tag{4}$$

We are interested in $P_k T_\varphi P_k$ not on the whole space H_k, but only on its subspace H_k^0. It is convenient to replace this subspace H_k^0 by another subspace isomorphic to it.

For this purpose we assume that the functions $f(g)$, $g \in F_k$, are extended to the set ΔF_k by the formula $f(\gamma g) = f(g)$ for arbitrary $\gamma \in \Delta$ and $g \in F_k$.

We consider the map

$$Q : f(g) \to \mu^{-1} \int_{\Delta \setminus Z} f(zg)\, dz \tag{5}$$

of H_k into itself, where μ is the measure of $\Delta \setminus Z$. Clearly the kernel of this map is our subspace H_k^0, and the image is the subspace H_k^1 of all functions in H_k satisfying the condition

$$f(zg) = f(g). \tag{6}$$

From this it follows that H_k^0 is isomorphic to the orthogonal complement $\tilde{H}_k$ in H_k^1.

So we may replace H_k^0 by $\tilde{H}_k$, the orthogonal complement to the subspace of functions $f(g) \in H_k$ satisfying the condition (6).

We show that the trace of $P_k T_\varphi P_k$ in $\tilde{H}_k$ is finite. In other words, we have to show that the trace of the operator:

$$P_k T_\varphi P_k - Q P_k T_\varphi P_k Q, \tag{7}$$

where Q is the projection operator onto H_k^1 given by (5), is finite.

Let us find the kernel of this operator. By (5) $QP_k T_\varphi P_k Q$ is given by the kernel

$$K_1(g_1, g_2) = \mu^{-2} \int_{\Delta \setminus Z} \int_{\Delta \setminus Z} K(z_1 g_1, z_2 g_2)\, dz_1\, dz_2, \tag{8}$$

where K is the kernel of T_φ. Consequently, the kernel of

$P_k T_\varphi P_k - Q P_k T_\varphi P_k Q$ has the form

$$K(g_1, g_2) - \mu^{-2} \int_{\Delta\backslash Z} \int_{\Delta\backslash Z} K(z_1 g_1, z_2 g_2)\, dz_1\, dz_2. \tag{9}$$

We have to show that this operator has a finite trace, that is, that the following integral converges:

$$I = \int_{F_k} \left[K(g, g) - \mu^{-2} \int_{\Delta\backslash Z} \int_{\Delta\backslash Z} K(z_1 g, z_2 g)\, dz_1\, dz_2 \right] dg, \tag{10}$$

Let us transform this integral.

Substituting in I the explicit expression (4) for the kernel K we obtain†

$$I = \int \sum_{\gamma\in\Gamma} \Big[\varphi(u^{-1}a^{-1}z^{-1}\gamma zau) - \mu^{-2} \int_{\Delta\backslash Z} \int_{\Delta\backslash Z} \varphi(u^{-1}a^{-1}z^{-1}z_1^{-1}\gamma z_2 zau)\, dz_1\, dz_2 \Big] \cdot \alpha\, d\alpha\, dz\, du. \tag{11}$$

We simplify this expression. First, when we replace the function φ by its average $\varphi_1(g) = \int \varphi(u^{-1}gu)\, du$ over the subgroup U of orthogonal matrices, we may write

$$I = \int \sum_{\gamma\in\Gamma} \Big[\varphi_1(a^{-1}z^{-1}\gamma za) - \mu^{-2} \int_{\Delta\backslash Z} \int_{\Delta\backslash Z} \varphi_1(a^{-1}z^{-1}z_1^{-1}\gamma z_2 za)\, dz_1\, dz_2 \Big] \alpha\, d\alpha\, dz. \tag{12}$$

Now we show that the summation in (12) is, in fact, taken only over the elements $\gamma \in \Delta = \Gamma \cap Z$. In other words, we prove the following proposition: *if* $\varphi(a^{-1}z_1^{-1}\gamma z_2 a) \neq 0$, *where* $z_1, z_2 \in Z$ *and* $a = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $0 < \alpha < N$, *then* $\gamma \in \Delta$.

As a preliminary we recall that the function $\varphi_1(g)$ is assumed to be equal to zero outside a sufficiently small neighborhood V of the unit element.

Suppose then that $\varphi_1(a^{-1}z_1^{-1}\gamma z_2 a) \neq 0$, that is,

$$a^{-1}z_1^{-1}\gamma z_2 a = v \in V. \tag{13}$$

We represent v in the form

$$v = za'u, \tag{14}$$

† Here we have used a formula for the invariant measure on g in terms of the parameters z, a, and u: if $g = zau$, then $dg = \alpha\, d\alpha\, dz\, du$, where dz and du are the invariant measures on Z and on U.

where $z \in Z$, u is an orthogonal matrix, and

$$a' = \begin{pmatrix} \alpha' & 0 \\ 0 & \alpha'^{-1} \end{pmatrix}.$$

It is easy to see that if V is a sufficiently small neighborhood of the unit element, then the element α' of a' is arbitrarily near to the unit element.

From (13) and (14) we obtain that

$$\gamma z_2 a = z' a a' u. \tag{15}$$

Consequently, the horospheres $x_z = x_0 z a$ and $x_z = x_0 z a a' u$ in X have a point in common.

Now we recall that the set of points in X of the form

$$x = x_0 z a u, \tag{16}$$

where z ranges over Z, u over the orthogonal matrices, and a over the diagonal matrices $a = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $0 < \alpha < N$, forms a cylindrical set in X. It is easy to check that for $0 < \alpha < N + \varepsilon$, where $\varepsilon > 0$ is sufficiently small, the elements (16) still form a cylindrical set.

Consequently, from the fact that the horospheres $x_z = x_0 z a$ and $x_z = x_0 z a a' u$ have a common point of intersection it follows that they coincide entirely and that $a' = 1$ and $u = 1$. But then it follows from (15) that $\gamma \in \Gamma \cap Z$.

So we have shown that the summation in (12) is, in fact, over the elements $\gamma \in \Delta = \Gamma \cap Z$. Hence this expression may be rewritten in the following form:

$$I = \int \sum_{\gamma \in \Delta} \Bigg[\varphi_1(a^{-1} z^{-1} \gamma z a) - \mu^{-2} \int_{\Delta\backslash Z} \int_{\Delta\backslash Z} \varphi_1(a^{-1} z^{-1} z_1^{-1} \gamma z_2 z a)\, dz_1\, dz_2 \Bigg] \alpha\, d\alpha\, dz. \tag{17}$$

Since Z is a commutative group, the expression under the integral sign does not depend on z. Therefore, on integrating with respect to z we obtain

$$I = \mu \int \sum_{\gamma \in \Delta} \Bigg[\varphi_1(a^{-1} \gamma a) - \mu^{-1} \int_{\Delta\backslash Z} \varphi_1(a^{-1} \gamma z a)\, dz \Bigg] \alpha\, d\alpha. \tag{18}$$

In this expression we go over from the matrix to the elements.

Δ is an infinite cyclic group. Thus, the elements $\gamma \in \Delta$ have the form

$$\gamma = \begin{pmatrix} 1 & 0 \\ n\sigma & 1 \end{pmatrix},$$

where σ is fixed and n ranges over the integers. For simplicity we take $\sigma = 1$.

Then we have

$$a^{-1}\gamma a = \begin{pmatrix} 1 & 0 \\ n\alpha^2 & 1 \end{pmatrix}, \qquad a^{-1}\gamma z a = \begin{pmatrix} 1 & 0 \\ (n+z)\alpha^2 & 1 \end{pmatrix}.$$

We introduce the function of a single variable

$$\psi(x) = \varphi_1(z), \tag{19}$$

where $z = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$. Then the expression (18) can be rewritten in the form:

$$I = \int_0^N \sum_{n=-\infty}^{+\infty} \left[\psi(n\alpha^2) - \int_0^1 \psi((n+z)\alpha^2)\, dz \right] \alpha\, d\alpha, \tag{20}$$

where $\psi(x)$ is a finite infinitely differentiable function.

So we have to show that the integral (20) converges. Since

$$\int_0^1 \psi((n+z)\alpha^2)\, dz = \psi((n+\theta_n)\alpha^2), \qquad 0 \le \theta_n \le 1,$$

we have

$$\psi(n\alpha^2) - \int_0^1 \psi((n+z)\alpha^2)\, dz = -\psi'((n+\theta_n')\alpha^2)\theta_n\alpha^2,$$

where $0 \le \theta_n \le 1$, $0 \le \theta_n' \le 1$.

So the integral (20) is majorized by

$$I_1 = \int_0^N \sum_{n=-\infty}^{+\infty} |\psi'((n+\theta_n')\alpha^2)|\, \alpha^3\, d\alpha. \tag{21}$$

Since $\psi'(x)$ is a finite function, the summation in (21) is, in fact, only over those n for which $|n+\theta_n'|\alpha^2 < C$. where C is a constant. Consequently,

$$\sum_{n=-\infty}^{+\infty} |\psi'((n+\theta_n')\alpha^2)| \le \frac{C_1}{\alpha^2},$$

therefore,

$$I_1 \le \int_0^N C_1\alpha\, d\alpha < \infty.$$

Thus, we have proved that the trace of $P_k T_\varphi P_k$ in H_k^0 is finite.

APPENDIX TO CHAPTER 1

Arithmetic Subgroups of the Group G of Real Unimodular Matrices of Order 2

1. Definition of an Arithmetic Subgroup. Here we discuss examples of discrete subgroups of the group G of real unimodular matrices of order 2. Among all discrete subgroups of G the most interesting and most important ones are the arithmetic subgroups, which we now define.

Let $g \to T(g)$ be any finite-dimensional representation of G. We examine the set of all elements $g \in G$ that correspond to integral matrices $T(g)$. It is not hard to verify that these elements g form a discrete subgroup of G. All discrete groups so obtainable, and also all their subgroups of finite index, are called arithmetic subgroups. The simplest example of an arithmetic subgroup of G is the group Γ of all *integral* matrices

$$\gamma = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \qquad m_{11}m_{22} - m_{12}m_{21} = 1.$$

It is called the *modular* group.

In **4** we shall give other examples of discrete subgroups—the so-called quarternion groups. From results of A. Weil it follows that the modular and quarternion groups and their subgroups of finite index exhaust all arithmetic subgroups of G.

Our definition of an arithmetic subgroup differs somewhat from the usual one. We now give the usual definition of an arithmetic subgroup of an arbitrary semisimple Lie group.

First we introduce the concept of a linear algebraic group.

We consider the group of all nonsingular matrices of order n over the field of complex numbers and a certain finite set of polynomial relations among the elements of the matrices. We single out the collection of all matrices that satisfy these relations. If this collection of matrices forms a group, then it is called a *linear algebraic group*.

If the coefficients of the polynomials belong to the field of rational numbers, we say that *the group is defined over the field of rational numbers*. We denote this group by G, and understand by G not so much the set of points, but the set of polynomial relations.

If k is any commutative ring over the field of rational numbers, we denote by G_k the set of matrices with elements from k, satisfying these relations and having as their determinant the unit element of the ring.

An arithmetic subgroup of a semisimple Lie group G_R (R is the field of real numbers) is any discrete subgroup obtainable by the following construction.

Let $G'_R \supset G_R$ be an arbitrary semisimple Lie group that contains G_R as a subgroup and is the direct product

$$G'_R = G_R \cdot K$$

of G_R and a compact group K. In G'_R we choose an arbitrary discrete subgroup Γ' that is commensurable with G'_Z, where Z is the ring of integers. (This means that $\Gamma' \cap G'_Z$ is of finite index in G'_Z and in Γ'.) Let $\Gamma \cap G_R$ be the image of Γ' under the natural mapping $G'_R \to G_R$. All subgroups Γ so obtainable are called arithmetic subgroups of G_R.

2. The Modular Group. In this subsection we construct a fundamental domain of the modular group Γ and show that this domain has finite volume.

From **§ 1.2** we know that the homogeneous space $X = \Gamma \backslash G$ can be interpreted as the space of linear elements of a certain Riemann surface $\mathscr{D}$.

Our next task is to describe this Riemann surface. We recall that according to **§ 1.2** the surface $\mathscr{D}$ is constructed as follows. On the half plane $\operatorname{Im} z > 0$ we consider all linear fractional transformations corresponding to elements of Γ. By identifying points that can be carried one into the other by these transformations we obtain the required surface $\mathscr{D}$.

To describe this Riemann surface explicitly, we construct a fundamental domain of Γ on the half-plane $\operatorname{Im} z > 0$, where this group acts as a group of fractional-linear transformations.

On the half-plane we consider the domain $\mathscr{D}$ given by the following inequalities:

$$|z| > 1, \qquad |\operatorname{Re} z| < \tfrac{1}{2}. \tag{1}$$

or, what is equivalent, by

$$|z| > 1, \qquad |z + 1| > |z|, \qquad |z - 1| > |z| \tag{2}$$

(Figure 3). We show that this is a fundamental domain for the modular group Γ.

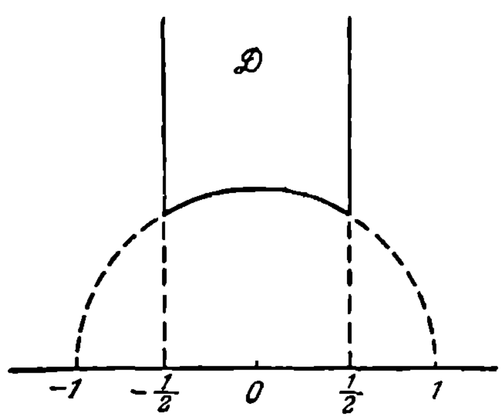

Figure 3.

To begin with, we show that every point of the half-plane can be carried into $\overline{\mathscr{D}}$ by some transformation from Γ.

Let z be an arbitrary point of the half-plane $\operatorname{Im} z > 0$. We consider the plane lattice formed by the points

$$w = mz + n,$$

where m and n range over all integers. From the points of the lattice we select one that is closest to zero (in the sense of the ordinary Euclidean distance). Let this be

$$w_1 = m_{12}z + m_{22}.$$

Next we inspect the points of the lattice that do not lie on the line through 0 and w_1, and from among them we again select a point closest to 0. Let this be

$$w_2 = m_{11}z + m_{21}.$$

By the definition of w_1 and w_2, the triangle with the vertices 0, w_1, w_2 contains no point of the lattice other than the vertices. Hence it follows easily that the parallelogram with the vertices 0, w_1, w_2, $w_1 + w_2$ also contains no point of the lattice other than the vertices (Figure 4).†

We show that

$$m_{11}m_{22} - m_{12}m_{21} = \pm 1.$$

For this purpose it is sufficient to verify that every lattice point w, and among them 1 and z, is an integral linear combination of w_1 and w_2. We represent w in the form

$$w = a_1w_1 + a_2w_2,$$

where a_1 and a_2 are real numbers. We have to show that then a_1 and a_2 are integers. If we represent these numbers in the form

$$a_1 = m_1 + r_1, \qquad a_2 = m_2 + r_2,$$

where m_1 and m_2 are integers and $0 \le r_1, r_2 < 1$, it is clear that the point

$$w' = w - m_1w_1 - m_2w_2 = r_1w_1 + r_2w_2$$

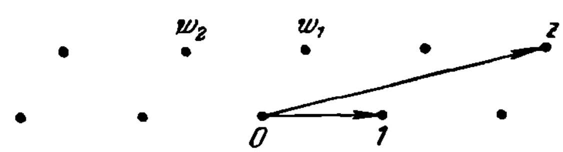

Figure 4.

† For if the triangle with the vertices w_1, w_2, $w_1 + w_2$ contains another lattice point w, then there is also a lattice point in the triangle with the vertices 0, w_1, w_2, namely $w_1 + w_2 - w$.

also is a point of our lattice. But this point belongs to the parallelogram with the vertices 0, w_1, w_2, $w_1 + w_2$. Consequently $w' = 0$, that is, $w = m_1 w_1 + m_2 w_2$.

So we have shown that $m_{11}m_{22} - m_{12}m_{21} = \pm 1$. By changing, if necessary, the signs of m_{12} and m_{22} we may assume that

$$m_{11}m_{22} - m_{12}m_{21} = 1.$$

Now we consider the point

$$z' = \frac{w_2}{w_1} = \frac{m_{11}z + m_{21}}{m_{12}z + m_{22}}.$$

We show that it belongs to $\overline{\mathscr{D}}$. For from the definition of w_1 and w_2 it follows that

$$|w_2| \geq |w_1|, \qquad |w_2 + w_1| \geq |w_2|, \qquad |w_2 - w_1| \geq |w_2|.$$

Dividing all these inequalities by $|w_1|$ we find that

$$|z'| \geq 1, \qquad |z' + 1| \geq |z'|, \qquad |z' - 1| \geq |z'|,$$

that is, z' belongs to $\overline{\mathscr{D}}$.

So we have shown that every point of the half-plane $\operatorname{Im} z > 0$ can be carried by transformations of Γ into the closure $\overline{\mathscr{D}}$ of $\mathscr{D}$.

Now let us see what points of $\overline{\mathscr{D}}$ can be carried into one another by transformations of Γ. The only such point pairs are on the boundary of $\overline{\mathscr{D}}$ and symmetrical with respect to the imaginary axis; this shows that $\mathscr{D}$ is a fundamental domain.

Suppose that the point $z_1 = x_1 + iy_1$ of $\overline{\mathscr{D}}$ is carried into another point $z_2 = x_2 + iy_2$ of $\overline{\mathscr{D}}$ by a transformation γ from Γ:

$$z_2 = \frac{m_{11}z_1 + m_{21}}{m_{12}z_1 + m_{22}}. \tag{3}$$

Our object is to show that z_1 and z_2 lie on the boundary of $\overline{\mathscr{D}}$ and that $y_1 = y_2$.

Without loss of generality we may assume that $y_2 \geq y_1$. First we show that $y_2 \leq y_1$, whence $y_2 = y_1$. To do this, we use the equation

$$y_2 = \frac{y_1}{|m_{12}z + m_{22}|^2}, \tag{4}$$

which follows immediately from (3). Next we show that $|m_{12}z_1 + m_{22}| \geq 1$. For we have

$$|m_{12}z_1 + m_{22}|^2 = m_{12}^2(x_1^2 + y_1^2) + 2m_{12}m_{22}x_1 + m_{22}^2.$$

Since $x_1^2 + y_1^2 \geq 1$ and $|x_1| \leq \frac{1}{2}$ for points z_1 of $\overline{\mathscr{D}_1}$, we find that

$$|m_{12}z_1 + m_{22}|^2 \geq m_{12}^2 - |m_{12}m_{22}| + m_{22}^2 \geq 1$$

(excluding the point $m_{12} = m_{22} = 0$, which in fact cannot occur, since $m_{11}m_{22} - m_{12}m_{21} = 1$).

So $|m_{12}z_1 + m_{22}|^2 \geq 1$. By (4) this shows that $y_2 \leq y_1$ and consequently $y_1 = y_2$.

Now we show that the points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ lie on the boundary of $\mathscr{D}$. For since $y_1 = y_2$, we have by (4)

$$(m_{12}x_1 + m_{22})^2 + m_{12}^2 y_1^2 = 1.$$

By taking again all possible values of m_{12}, m_{22}, x_1, x_2 we easily see that this equation holds only in the following three cases:

1. $m_{12} = \pm 1, \quad m_{22} = 0, \quad x_1^2 + y_1^2 = 1,$
2. $m_{12} = \pm 1$, $m_{22} = \pm 1$, $x_1 = \pm \frac{1}{2}$ (the sign of x_1 is opposite to the sign of $m_{12}m_{22}$), $y_1 = \dfrac{\sqrt{3}}{2}$,
3. $m_{12} = 0, \quad m_{22} = \pm 1.$

In the first two cases z_1 lies on the circle $|z| = 1$, that is, on the boundary of $\mathscr{D}$. In the third case we must also take $m_{11} = \pm 1$, and therefore z_2 can be expressed in terms of z_1 as follows:

$$z_2 = z_1 + n,$$

where n is an integer ($n \neq 0$). But then $|z_2 - z_1| \geq 1$. Obviously this is possible only when z_1 and z_2 lie on the vertical parts of the boundary of $\mathscr{D}$. So we have shown that the domain $\mathscr{D}$ sketched in Figure 3 is in fact a fundamental domain for the modular group Γ.

Incidentally we have described the Riemann surface associated with the modular group Γ: it is the closure $\overline{\mathscr{D}}$ of $\mathscr{D}$, where points on the boundary of $\mathscr{D}$ that are symmetrical with respect to the imaginary axis are to be identified. Hence, this Riemann surface is homeomorphic to the sphere with one point deleted (corresponding to the point at infinity on $\mathscr{D}$).

Now we compute the area of this unbounded domain $\mathscr{D}$ and show that it is finite.

By definition, the element of area dv on the half-plane $\operatorname{Im} z > 0$ must be preserved under conformal transformations. From this condition we obtain easily that to within a constant factor the invariant element of area dv on the half-plane $\operatorname{Im} z > 0$ can be expressed by the following formula:

$$dv = \frac{dx\,dy}{y^2}.$$

Consequently, the area $S(\mathscr{D})$ of $\mathscr{D}$ is expressed by the formula

$$S(\mathscr{D}) = \iint\limits_{\mathscr{D}} \frac{dx\,dy}{y^2}.$$

Computing this integral we find that

$$S(\mathscr{D}) = \frac{\pi^2}{3}.$$

So we have shown that the fundamental domain $\mathscr{D}$ of Γ of the half-plane $\operatorname{Im} z > 0$ has finite area.

From this it follows immediately that on G a fundamental domain of Γ also has finite volume. For this fundamental domain may be realized as the space of linear elements on $\mathscr{D}$.

3. Some Subgroups of the Modular Group. In this subsection we investigate some important classes of subgroups of the modular group of finite index.

Let n be a fixed natural number, $n > 1$. We consider the ring Z_n of residue classes modulo n. We denote the residue class of a given integer a by a^*.

By Γ_n^* we denote the group of all unimodular matrices

$$\gamma^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

with elements from Z_n.

We have a natural homomorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \tag{1}$$

of Γ into Γ_n^*. The kernel Γ_n' of this homomorphism is called the principal congruence subgroup of degree n. Clearly Γ_n' consists of all integral unimodular matrices γ that are representable in the form

$$\gamma = e + n\gamma',$$

where e is the unit matrix, and γ' an integral matrix.

We show that *the map* (1) *is onto the whole group* Γ_n^*, *and consequently*:

$$\Gamma/\Gamma_n' \approx \Gamma_n^*.$$

Proof. Let $\gamma^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$ be any matrix from Γ_n^*, where a, b, c, d are arbitrarily chosen elements from the corresponding residue classes a^*, b^*, c^*, d^*. Then we have $ad - bc \equiv 1 \pmod{n}$, that is $ad - bc = 1 + mn$, where m is an integer. Obviously the greatest common divisor (c, d) of c and d is prime to n. Therefore we can find a q for which c and $d + qn$ are relatively prime. Without loss of generality we may assume that $(c, d) = 1$.

We consider the matrix

$$\gamma = \begin{pmatrix} a + rn & b + sn \\ c & d \end{pmatrix}.$$

Its determinant is $ad - bc + n(rd - sc) = 1 + n(m - rd - sc)$. Since d and c are relatively prime, we can select integers r and s such that $m - rd - sc = 0$, that is, that γ becomes a unimodular matrix. So we have shown that every matrix $\gamma^* \in \Gamma_n^*$ has an inverse image in Γ.

We compute the index $\Gamma : \Gamma_n'$ of Γ_n' or, what is equivalent, the order $|\Gamma_n^*|$ of Γ_n^*. We show that

$$|\Gamma_n^*| = n^3 \Pi \left(1 - \frac{1}{p^3}\right), \tag{2}$$

where the product is taken over the distinct prime divisors p of n.

Let p be a prime divisor of n. Then there exists a unique homomorphism $Z_n \to Z_{n/p}$. This homomorphism induces a homomorphism of the corresponding groups

$$\Gamma_n^* \to \Gamma_{n/p}^*.$$

We denote by $I_{n,p}$ the kernel of this homomorphism. Then we have

$$|\Gamma_n^*| = |I_{n,p}|\,|\Gamma_{n/p}^*|.$$

Therefore, if we can find the order $|I_{n,p}|$ of $I_{n,p}$, then by an elementary induction with respect to the number of prime factors of n we can compute the order $|\Gamma_n^*|$ of Γ_n^*.

Hence, we compute the order of $I_{n,p}$. Obviously, the group of matrices $I_{n,p}$ consists of all matrices $\gamma^* \in \Gamma_n^*$ that are representable in the form

$$\gamma^* = e + \frac{n}{p}\gamma_1^*.$$

In other words, the elements of $I_{n,p}$ are the matrices of the form

$$\begin{pmatrix} 1 + \frac{n}{p}a & \frac{n}{p}b \\ \frac{n}{p}c & 1 + \frac{n}{p}d \end{pmatrix},$$

where a, b, c, d are elements of the residue class ring modulo p. By hypothesis, the determinant of these matrices is congruent to 1 (mod n), that is,

$$1 + \frac{n}{p}(a + d) + \frac{n^2}{p^2}(ad - bc) \equiv 1 \pmod{n}.$$

When we subtract 1 and cancel the factor n/p, we find

$$a + d + \frac{n}{p}(ad - bc) \equiv 0 \pmod{p}. \tag{3}$$

The order of $I_{n,p}$ is equal to the number of solutions of this congruence. We consider the two possible cases:

1. n/p is divisible by p. In this case (3) assumes the form

$$a + d \equiv 0 \pmod{p}.$$

Thus, a, b, c may be arbitrary residue classes modulo p, and the element d is uniquely expressed in terms of a. Consequently, the order of $I_{n,p}$ is equal to p^3.

2. The numbers p and n/p are relatively prime. In this case we write (3) in the form

$$a\left(1 + \frac{n}{p} d\right) + d - \frac{n}{p} bc \equiv 0 \pmod{p}.$$

If $1 + \frac{n}{p} d \not\equiv 0 \pmod{p}$, the elements b and c may be arbitrary, and a is uniquely expressible in terms of b, c, d. Consequently, the number of elements of $I_{n,p}$ satisfying the condition $1 + \frac{n}{p} d \not\equiv 0$ is equal to $p^2(p - 1)$. But if $1 + \frac{n}{p} d \equiv 0 \pmod{p}$, then the product bc has a fixed nonzero value, and a is arbitrary. Therefore, the number of elements of $I_{n,p}$ satisfying the condition $1 + \frac{n}{p} d \equiv 0 \pmod{p}$ is equal to $(p - 1)p$.

Thus, the total number of elements of $I_{n,p}$ is equal to

$$(p - 1)p^2 + (p - 1)p = p^3\left(1 - \frac{1}{p^2}\right).$$

So we have established that

$$|I_{p,n}| = \begin{cases} p^3 & \text{if } \frac{n}{p} \text{ is divisible by } p, \\ p^3\left(1 - \frac{1}{p^2}\right) & \text{if } \frac{n}{p} \text{ is not divisible by } p. \end{cases}$$

From the equation $|\Gamma_n^*| = |I_{n,p}|\,|\Gamma_{n/p}^*|$ we obtain immediately, by induction over the number of prime factors of n, the required formula

$$|\Gamma_n^*| = n^3\Pi\left(1 - \frac{1}{p^2}\right),$$

where the product is taken over the distinct prime factors p of n.

The result we have obtained can be used to compute the area v of a fundamental domain on the half-plane $\operatorname{Im} z > 0$ relative to the subgroup Γ'_n.

We make use of the following obvious remarks. Let Γ' be a discrete subgroup of G and Γ'' a subgroup of finite index in Γ'. Then if F' is a fundamental domain relative to Γ', a fundamental domain for Γ'' is the union

$$F'' = \bigcup_{\gamma} \gamma F'$$

of the sets $\gamma F'$, where γ ranges over one representative each from every coset $\Gamma'' \setminus \Gamma'$.

Hence, it follows that the areas $v_{\Gamma'}$ and $v_{\Gamma''}$ of the fundamental domains of the subgroups Γ' and Γ'' are connected by the relation

$$v_{\Gamma''} = [\Gamma' : \Gamma'']v_{\Gamma'}.$$

where $[\Gamma' : \Gamma'']$ is the index of Γ'' in Γ'.

In **2** we have established that the area of a fundamental domain relative to the modular group Γ is equal to $\pi^2/3$. Consequently, on the basis of (2) we conclude: the area $v_{\Gamma'_n}$ of a fundamental domain relative to the congruence subgroup Γ'_n is equal to

$$v_{\Gamma'_n} = \frac{\pi^2}{3} n^3 \Pi \left(1 - \frac{1}{p^2}\right)$$

(the product is taken over all prime divisors p of n).

Now we indicate another class of subgroups of the modular group Γ. We denote by Γ_n the set of matrices of Γ of the form

$$\begin{pmatrix} a & nb \\ nc & d \end{pmatrix},$$

where a, b, c, d are integers. It is clear that Γ_n is a group and that $\Gamma_n \supset \Gamma'_n$.

Let us compute the index $\Gamma : \Gamma_n$ of Γ_n in Γ. For this purpose we note that the factor group Γ_n / Γ'_n is isomorphic to the group of all diagonal unimodular matrices

$$\gamma^* = \begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix},$$

where a^* and d^* are elements of the residue class ring modulo n. Obviously, the number of such matrices is equal to the number $\varphi(n)$ of natural numbers $x < n$ that are prime to n,

$$\varphi(n) = n \Pi \left(1 - \frac{1}{p}\right),$$

where the product is taken over all prime divisors p of n. Consequently we have

$$[\Gamma:\Gamma'_n] = n\Pi\left(1 - \frac{1}{p}\right).$$

But then

$$\Gamma:\Gamma_n = \frac{\Gamma:\Gamma'_n}{\Gamma_n:\Gamma'_n} = n^2\Pi\left(1 + \frac{1}{p}\right).$$

On the basis of this result we find that the area v_{Γ_n} of a fundamental domain relative to Γ_n is equal to

$$v_{\Gamma_n} = \frac{\pi^2}{3} n^2\Pi\left(1 + \frac{1}{p}\right),$$

where the product is taken over all prime divisors p of n.

Finally, we mention the subgroups $\hat{\Gamma}_n$ of the modular group Γ that consist of the matrices of the form

$$\begin{pmatrix} a & nb \\ c & d \end{pmatrix},$$

where a, b, c, d are integers. By arguments similar to the above we can easily verify that

$$\Gamma:\hat{\Gamma}_n = n\Pi\left(1 + \frac{1}{p}\right),$$

where, as before, the product is taken over all prime divisors p of n.

4. Quaternion Groups. In **2** and **3** we have chosen examples of arithmetic subgroups of Γ for which the space $X = \Gamma \setminus G$ has finite volume, but is not compact. In this subsection we construct another class of arithmetic subgroups of G—the so-called quarternion groups. We shall show that for these groups $X = \Gamma \setminus G$ is compact. We construct the quarternion groups with the help of certain algebraic notions.

We begin with the description of a class of algebras over the field of rational numbers.

We consider an algebra A over the field of rational numbers and with the basis 1, α, β, γ, where 1 denotes the unit element, and α, β, γ are connected by the relations

$$\gamma = \alpha\beta = -\beta\alpha, \qquad \alpha^2 = a, \qquad \beta^2 = b \tag{1}$$

a and b being positive integers. Thus, every element of A has the form

$$x = x_0 + x_1\alpha + x_2\beta + x_3\gamma, \tag{2}$$

where x_0, x_1, x_2, x_3 are rational numbers.

If A is a division algebra, it is often called an algebra of generalized quarternions or simply a quarternion algebra. We adopt this terminology here.

First of all we show that with every element x of A we can associate in a one-to-one fashion a real matrix g_x such that

$$g_x + g_y = g_{x+y}, \qquad g_{xy} = g_x g_y. \tag{3}$$

For we set

$$g_x = \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2\sqrt{b} + x_3\sqrt{ab} \\ x_2\sqrt{b} - x_3\sqrt{ab} & x_0 - x_1\sqrt{a} \end{pmatrix}. \tag{4}$$

The verification of the relations (3) is left to the reader.

The determinant of g_x is

$$x_0^2 - x_1^2 a - x_2^2 b + x_3^2 ab. \tag{5}$$

The expression (5) is usually called the *norm* of x and is denoted by $N(x)$. Obviously,

$$N(xy) = N(x)N(y), \qquad N(1) = 1.$$

The role of the norm $N(X)$ is clear from the following theorem. *If $N(x) = 0$ for $x = 0$ only, then A is a division algebra. Conversely, if A is a division algebra, then $N(x) = 0$ for $x = 0$ only.*

Proof. Suppose that $N(x) \neq 0$. Then the element

$$x^{-1} = \frac{1}{N(x)}(x_0 - x_1\alpha - x_2\beta - x_3\gamma)$$

is easily seen to be inverse to x. Conversely, if A is a division algebra, then $N(x)N(x^{-1}) = 1$ and hence $N(x) \neq 0$.

We give an example of a division algebra. Let b be a prime number and a an arbitrary number that is a quadratic nonresidue modulo b (that is, the congruence $x^2 = a \pmod{b}$ has no solution in integers). We show that then the algebra A defined by the relations (1) is a division algebra.

For otherwise there exists an element $x \neq 0$ of A with the norm

$$x_0^2 - ax_1^2 - bx_2^2 + abx_3^2 = 0. \tag{6}$$

Without loss of generality we may assume that x_0, x_1, x_2, x_3 are integers without common divisor. From (6) it follows that $x_0^2 \equiv ax_1^2 \pmod{b}$; consequently, since a is a quadratic nonresidue modulo b, the integers x_0 and x_1 must be divisible by b. But then again it follows from (6) that $x_2^2 \equiv ax_3^2 \pmod{b}$ so that x_2 and x_3 are also divisible by b. This contradicts the assumption that the integers x_0, x_1, x_2, x_3 have no common divisor.

Now we pass on to the definition of a quarternion group. Let A be a quarternion division algebra. We consider the set Γ of matrices g_x with determinant 1 for which x_0, x_1, x_2, x_3 are integers. Γ is obviously a group, and we show that Γ is discrete.

For this purpose it is sufficient to indicate a neighborhood of the unit element of the group G of all real unimodular matrices of order 2 in which there are no elements of Γ other than the unit element. Such a neighborhood is, for example, the set of matrices of the form

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where

$$|g_{11} - 1| < \tfrac{1}{2}, \quad |g_{12}| < \tfrac{1}{2}, \quad |g_{21}| < \tfrac{1}{2}, \quad |g_{22} - 1| < \tfrac{1}{2}.$$

For suppose that this neighborhood contains the matrix $g_x \in \Gamma$, that is, the matrix with the elements

$$g_{11} = x_0 + x_1\sqrt{a}, \qquad g_{12} = x_2\sqrt{b} + x_3\sqrt{ab},$$
$$g_{21} = x_2\sqrt{b} - x_3\sqrt{ab}, \qquad g_{22} = x_0 - x_1\sqrt{a},$$

where x_0, x_1, x_2, x_3 are integers. From the inequalities it follows that

$$|g_{11} + g_{22} - 2| < 1, \qquad |g_{12} + g_{21}| < 1,$$

that is,

$$|2x_0 - 2| < 1, \qquad |2x_2\sqrt{b}| < 1.$$

Consequently, $x_0 = 1$, $x_2 = 0$. Next, from the inequalities

$$|g_{11} - 1| < \tfrac{1}{2}, \qquad |g_{12}| < \tfrac{1}{2}$$

we find that $x_1 = x_3 = 0$. Thus, g_x is the unit matrix.

Now we show that *the factor space* $\Gamma \setminus G$ *is compact.* First we show that for every matrix g with determinant 1 there exists a matrix g_x with integral x_0, x_1, x_2, x_3, but not necessarily with determinant 1, such that $g_x g$ belongs to some fixed compact domain.

We note that for a fixed matrix g the elements of $g_x g$ are linear forms l_{ij} in x_0, x_1, x_2, x_3. It is not hard to compute the determinant of this system of linear forms: it is equal to $4ab$. Consequently, by Minkowski's lemma,† there exist integers x_0, x_1, x_2, x_3, not all zero, such that $|l_{ij}| \leq c_{ij}$, where c_{ij} are arbitrarily chosen positive constants whose product is equal to $4ab$.

We denote by F the set of real matrices g for which $|g_{ij}| \leq c_{ij}$. We have shown that for every matrix g with determinant 1 there exists a matrix g_x, where x is an integral quarternion, such that $g_x g \in F$.

† The statement and proof of the Minkowski's lemma will be given at the end of this Appendix on p. 118.

The proposition we need comes out when we apply Minkowski's lemma to the parallelepiped $|l_{ij}| \leq c_{ij}$, $i, j = 1, 2$, in four-dimensional space.

The set F need not be compact. However, we shall show presently that $g_x g$ lies, in fact, in a certain compact subset of F.

We have $\det(g_x g) = \det g_x = N(x)$. Now we use the fact that A is a division algebra so that $\det g_x = N(x) \neq 0$. Furthermore since $N(x)$ is an integer, $\det(g_x g)$ is also an integer, different from zero. We denote by F_m, $m \neq 0$, the set of elements $g \in F$ with determinant m. It is clear that F_m is compact and that for sufficiently large $|m|$ the set F_m is empty. By what we have shown, $g_x g$ lies in the union of the sets F_m, $m \neq 0$, that is, in a compact set.

So we have shown that for every unimodular matrix g there exists an integral, but not necessarily unimodular, matrix g_x such that $g_x g$ lies in a compact set.

Let us call integral quarternions x and y equivalent if xy^{-1} is a quarternion integer of norm 1.

To complete the proof that $\Gamma \setminus G$ is compact we prove the following lemma.

LEMMA. *The set of integral quarternions with norm m consists of a finite number of classes of equivalent quarternions.*

Proof. With every integral quarternion x $(N(x) = m)$ we associate the matrix a_x of order four that expresses the transformation $y \to yx$ in the basis (1). It is easy to verify that a_x is an integral matrix with determinant m^2.

It is well known that among integral matrices of order n with a given value of the determinant Δ there exists only a finite number of matrices $a_1, \ldots, a_p$ such that every matrix with determinant Δ is of the form $a_k\alpha$, where α is a unimodular integral matrix.† Thus, among the matrices $a_{x_i}(N(x) = m)$ there exist matrices a_x $(N(x) = m)$ such that every matrix $a_{x_i}(N(x) = m)$ is equal to $a_{x_i}\alpha$, where α is an integral unimodular matrix. Since the map $x \to a_x$ carries the product of quarternions into the product of their corresponding matrices, we have $\alpha = a_i^{-1}x$. From the fact that α is integral it follows that the quarternion $x_i^{-1}x$ is integral. This completes the proof.

* * *

MINKOWSKI'S LEMMA. *Let a lattice in n-dimensional space be given, that is, a set of points $(l_1, \ldots, l_n)$, $l_i = \sum_{i=1}^{n} l_{ij}n_j$, where l_{ij} are fixed real numbers and n_j ranges over all integers: it is assumed that the determinant $|l_{ij}|_{i,j=1}^{n} = \Delta$ is different from zero. Then*

† This follows from the following easily verified statement: Every integral non-singular matrix a can be carried by multiplication by a suitable integral unimodular matrix α into the following form:

$$a\alpha = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where $|a_{ij}| < |a_{ii}|$ for $j < i$; $i, j = 1, \ldots, n$.

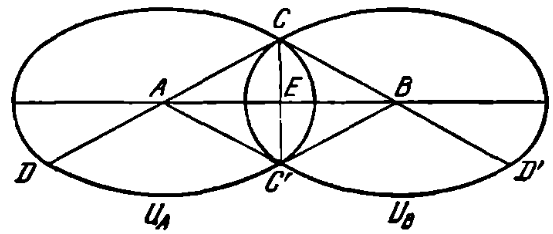

Figure 5.

every convex centrally-symmetric body with its center at O having a volume $v \geq 2^n\Delta$ *contains at least two points of this lattice (symmetric with respect to the set).*

Proof. Suppose that we have a convex body U with center at O and not containing any other points except O. We reduce this body linearly to half its size, by applying a similarity transformation with center at O, and denote the body so obtained by U_0. Next we construct bodies equal to U_0 and situated parallel to U_0 around all points of our lattice as centers. We show that the bodies so constructed have no common points.

For let us assume that two such bodies U_A and U_B with centers, respectively, CA and CB and construct the parallelogram $CAC'B$ (Figure 5). Let D be the point symmetric to C relative to A. This point belongs to U_A, (because A is a symmetry center of U_A). But then, since U_A and U_B are equal and parallel, C' must also belong to U_B. Since U_B is convex, the midpoint E of CC' also lies in U_B. Similarly we can see that E belongs to U_A. So we have shown that if two bodies with centers at the lattice points A and B have at least one point in common, then the midpoint of AB also is a common point. But this contradicts the assumption that the original body U contains no lattice points except its center.

Since these bodies with centers at the lattice points do not intersect, it is easy to see that their volumes are less than the volume Δ of the fundamental parallelepiped of the lattice. But then the volume of the original body U is less than 2^nV.

Thus, if a convex body with center at O contains no lattice points except O, its volume is less than $2^n\Delta$, where Δ is the volume of the fundamental parallelepiped of the lattice. Consequently, if a convex body U with center at O has volume $V \geq 2^n\Delta$, then it necessarily contains, apart from O, at least two lattice points.

REPRESENTATIONS OF THE GROUP OF UNIMODULAR MATRICES OF ORDER 2 WITH ELEMENTS FROM A LOCALLY COMPACT TOPOLOGICAL FIELD

2

In this chapter we study the representations of the group G of unimodular matrices of order 2 with elements from a locally compact topological field $\mathbf{K}$. The complete classification of all such fields is well known (see **§ 1**).

In **§§ 3** and **4** we construct the irreducible unitary representations of G.

The representation operators $T(g)$ are given by their kernels, which are generalized functions. The question is: what are the functions from which these kernels are formed?

Two types of functions on a locally compact field play the fundamental role—additive characters, which are generalizations of the exponential function, and multiplicative characters, which are generalizations of the power function.

An additive character on **K** is a continuous complex-valued function $\chi(x)$ satisfying the condition

$$\chi(x+y) = \chi(x)\chi(y)$$

for arbitrary elements x and y from **K**.

For the field of real numbers these functions have the form $\chi(x) = e^{\alpha x}$, where α is a complex number; for the field of complex numbers $z = x + iy$ they have the form $\chi(z) = e^{\alpha x+\beta y}$, where α and β are complex numbers.

A multiplicative character on **K** is a continuous complex-valued function $\pi(x)$ on $\mathbf{K} \setminus 0$ satisfying the condition

$$\pi(xy) = \pi(x)\pi(y)$$

for arbitrary nonzero elements x and y from **K**. For the field of real numbers these functions have the form $\pi(x) = |x|^\alpha$ or $\pi(x) = |x|^\alpha \operatorname{sign} x$, where α is an arbitrary complex number; for the field of complex numbers $z = re^{i\varphi}$ they have the form $\pi(z) = r^\alpha e^{in\varphi}$, where α is a complex and n a real number.

The entire stock of functions needed in the theory of representations (Gamma-functions, Beta-functions, Bessel functions, the hypergeometric function) are formed from additive and multiplicative characters by rational transformations of the independent variables and by integration with respect to parameters. In particular, we shall see in **§ 3** that the kernels of the operators of irreducible unitary representations of G can be expressed in terms of Bessel functions, or after transition to another basis in the representation space, by the hypergeometric function.

The group G has several series of irreducible unitary representations. One of these (the continuous series) is connected with the ground field **K**; each of the remaining (discrete) series is connected with a certain quadratic extension of **K**. Thus, if **K** is the field of complex numbers, there is only one series, because the field of complex numbers has no proper algebraic extensions; if **K** is the field of real numbers, there are two series of representations, because the field of real numbers has only one quadratic extension, and, if **K** is a disconnected field, then there are four series of representations because a disconnected field has three quadratic extensions.†

Within each series a representation is given by a certain multiplicative character. More accurately, a representation of the continuous series is given by a multiplicative character π on **K**, and

† Apart from certain special cases when the number of quadratic extensions of **K** is greater than three (see **§ 1**).

to the characters π and π^{-1} there correspond equivalent representations. A representation of the discrete series corresponding to the quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$ is given by a character on the unit circle in $\mathbf{K}(\sqrt{\tau})$, that is, on the multiplicative group of elements $t = x + \sqrt{\tau}y$ for which $t\bar{t} \equiv x^2 - \tau y^2 = 1$. Again, to the characters π and π^{-1} there correspond equivalent representations.

So there is complete duality between the irreducible representations of G and the Cartan subgroups of G: every irreducible representation of G is given by a character on one of the Cartan subgroups.

In the construction of the representations of the discrete series the following interesting fact emerges: these representations are realized not in the space of all functions on $\mathbf{K}$, but in a space of functions that resemble analytic functions. (For a disconnected field the concept of a complex-valued analytic function does not exist. Nevertheless there is a natural way of defining the concept of a function resembling an analytic function in the upper half-plane, see **§ 2.8.**)

In **§ 5** we compute the traces (characters) of the irreducible representations. We obtain a single formula for them, independent of the structure of $\mathbf{K}$. In fact, we shall see that the trace of the representation of the continuous series corresponding to the character $\pi(t)$ is expressed by the following formula:

$$\operatorname{Tr} T_\pi(g) = \int_K \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1})\pi(t)\, |t|^{-1}\, dt,$$

where λ_g and λ_g^{-1} are the eigenvalues of the matrix g, and $\delta(t)$ is the Delta-function.

It is convenient to combine the representations of the discrete series corresponding to a quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$ into pairs. Then the trace of the sum of the related representations of the discrete series is expressed by the following formula:

$$\operatorname{Tr} T_\pi(g) = 2 \int_{t\bar{t}=1} \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\, \pi(t)\, d^*t.$$

The meaning of the notation $|t|$ and $\operatorname{sign}_\tau t$ for a disconnected field will be explained in **§ 1.**

In **§ 6** we obtain the Plancherel formula, which gives the decomposition of the regular representation of G into representations of the continuous and the discrete series. Specifically, when we associate with each representation $T_\pi(g)$ of these series the operator

$$T_\pi(f) = \int f(g)\, T_\pi(g)\, dg,$$

where f is a function on G of integrable square, then we have the inversion formula

$$f(g) = \int \mu(\pi) \operatorname{Tr} (T_\pi(f) T_\pi^{-1}(g)) \, d\pi$$

and the Plancherel formula

$$\int |f(g)|^2 \, dg = \int \mu(\pi) \operatorname{Tr} (T_\pi(f) T_\pi^*(f)) \, d\pi.$$

It will be shown that the "Plancherel measure" $\mu(\pi)$ occurring in these formulae may be given by the following single formula:

$$\mu(\pi) = c \int \pi(t) \, |1 - t|^{-2} \, dt.$$

For representations of the continuous series the integration here is taken over **K**, and, for representations of the discrete series corresponding to a quadratic extension $\mathbf{K}(\sqrt{\tau})$ of **K**, over the unit circle $t\bar{t} \equiv x^2 - \tau y^2 = 1$. The integral must be understood in the sense of the regularizing value.

This integral can be computed without difficulty when **K** is the field of complex or real numbers.

For the field of complex numbers we find

$$\mu(\pi) = c(\rho^2 + n^2), \qquad \text{where } \pi(re^{i\varphi}) = r^{i\rho} e^{in\varphi}.$$

For the field of real numbers we have: for representations of the continuous series

$$\mu(\pi) = c\rho \tanh \frac{\pi\rho}{2}, \qquad \text{when } \pi(x) = |x|^{i\rho},$$

$$\mu(\pi) = c\rho \coth \frac{\pi\rho}{2}, \qquad \text{when } \pi(x) = |x|^{i\rho} \operatorname{sign} x;$$

for representations of the discrete series

$$\mu(\pi) = c\,|n|, \qquad \text{when } \pi(t) = t^n, \quad |t| = 1.$$

§ 1. STRUCTURE OF LOCALLY COMPACT FIELDS

In this section we give an account of essentially well-known results on the structure of locally compact fields. Some of the results will be only stated. Their detailed proof can be found, for example, in [8] and [61].

1. Classification of Locally Compact Fields. We discuss only topological fields (that is, fields with a nondiscrete topology).†

† Hence, the field of rational numbers is excluded from this discussion.

Here are the classical examples of locally compact topological fields:

1. The field R of real numbers.
2. The field C of complex numbers.
3. The field Q_p of p-adic numbers, where p is any prime number. Let us recall the definition of the field Q_p.

The elements of Q_p are formal power series

$$x = \sum_{i=k}^{+\infty} a_i p^i, \tag{1}$$

where k is an integer, and the a_i are integers satisfying the condition $0 \leq a_i < p$. Thus, the series (1) may contain an arbitrary finite number of terms with negative integral powers. The sum of the two p-adic numbers $x = \sum_{i=k}^{\infty} a_i p^k$ and $y = \sum_{i=l}^{\infty} b_i p^i$ is the p-adic number $z = \sum_{i=m}^{\infty} c_i p^i$, $m = \min(k, l)$ such that

$$\sum_{i=k}^{n} a_i p^i + \sum_{i=l}^{n} b_i p^i \equiv \sum_{i=m}^{n} c_i p^i (\operatorname{mod} p^{n+1}) \tag{2}$$

for every positive integer n. Obviously, the coefficients c_i can be found successively from the relations (2). The product of p-adic numbers is defined in a similar fashion. A neighborhood of a p-adic number $x = \sum_{i=k}^{\infty} a_i p^i$ is the set U_n of p-adic numbers $y = \sum_{i=k}^{\infty} b_i p^i$ for which $b_i = a_i$ for $i \leq n$. It is not hard to verify that under this topology Q_p becomes a locally compact space and that the operations of addition and multiplication are continuous in this topology.

We mention that the field Q_p can also be obtained by completing the field of rational numbers relative to a suitable topology.

For let $n(r)$ be the power to which the prime number p occurs as a factor in the rational number r. We call p^{-r} the p-norm of r. A sequence of rational numbers is called fundamental if it is fundamental in the sense of the p-norm.

Thus, Q_p contains the field of rational numbers as an everywhere dense subset.

4. The field $\mathbf{K}_p(t)$ of power series over the residue class field modulo p (p a prime number). By definition, the elements of $\mathbf{K}_p(t)$ are power series

$$x = \sum_{i=k}^{\infty} a_i t^i,$$

which may contain a finite number of terms with negative powers of t; the coefficients of these series lie in the residue class field modulo p. Addition and multiplication of two power series is defined in the natural way. A neighborhood of the power series $x = \sum_{i=k}^{\infty} a_i t^i$ is the

set of power series in which all the coefficients up to a certain fixed index coincide with the a_i.

Now we can give the classification of all locally compact nondiscrete fields (theorem of Koval'skii-Pontryagin).

The field R of real numbers and the field C of complex numbers are the only connected locally compact fields.

Every disconnected locally compact field of characteristic 0 *is a finite extension of the field* Q_p *of p-adic numbers.*

Every locally compact field of characteristic $p \neq 0$ *is a finite extension of the field* $K_p(t)$ *of power series over the residue class field modulo p.*

For fields of characteristic $p \neq 0$ there is an even stronger result. *Every locally compact field of characteristic* $p \neq 0$ *is isomorphic to the field of power series*

$$x = \sum_{i=k}^{\infty} a_i t_i,$$

whose coefficients belong to a finite field of characteristic p. The algebraic operations and the topology in this field are defined just as in the case of $\mathbf{K}_p(t)$.

2. The Norm in K. For an arbitrary locally compact field **K** we can introduce the concept of a norm. For this purpose we consider a measure dx on **K** invariant under addition:

$$d(x + a) = dx$$

for every a from **K**. Such a measure on **K** is known to be uniquely determined to within a constant factor.

Let x_0 be an arbitrary element from **K**. It is easy to see that the measure $d_{x_0}x = d(xx_0)$ is also invariant under addition, so that it differs from dx only by a factor depending on x_0, which we denote by $|x_0|$:

$$d_{x_0}x = |x_0|\, dx.$$

Thus, we have introduced in **K** a continuous function $|x|$, which obviously has the following properties:

1. $|x| > 0$ for $x \neq 0$; $|0| = 0$,
2. $|xy| = |x| \cdot |y|$.

It can be shown that for a disconnected field **K** also the following property holds:

3. $|x + y| \leq \max(|x|, |y|)$.

We call $|x|$ the *norm* of x in **K**.

Clearly, for the field of real numbers $|x|$ is the absolute value of x; for the field of complex numbers $|x|$ is the square of the modulus of x.

Let us see what values $|x|$, $x \neq 0$, may assume. For this purpose we observe that the map

$$x \to |x|$$

is a continuous homomorphism of the multiplicative group of $\mathbf{K}$ into the multiplicative group of positive real numbers. *From this it follows easily that for a connected field $|x|$ $(x \neq 0)$ ranges over all positive real numbers; but for a disconnected field $|x|$ $(x \neq 0)$ assumes only the discrete set of values q^n, where q is a fixed number and n is an integer.*

From the result just stated it follows that in a disconnected field $\mathbf{K}$ the set of points x for which $|x| = c$, $c > 0$, and the set of points x for which $|x| < c$, $c > 0$, are both open in $\mathbf{K}$.

It can be shown that the sets of points x of a disconnected field $\mathbf{K}$ for which $|x| < c$ (when c ranges over the positive numbers) form a complete system of neighborhoods of the zero element. Hence, *the topology in a disconnected field $\mathbf{K}$ is completely determined by the norm in* $\mathbf{K}$. For connected fields the last result is obvious.

3. The Structure of Disconnected Fields. Using the concept of norm we can describe the detailed structure of disconnected fields. Let $\mathbf{K}$ be a disconnected field with norm $|x|$. Then the following facts hold:

1. The set O of elements of $\mathbf{K}$ for which $|x| \leq 1$ is compact and open in $\mathbf{K}$. Obviously O is a subring whose elements are called the *integers* of $\mathbf{K}$.

2. The set of elements x in O for which $|x| < 1$ forms a prime ideal P of O. The residue class field $\mathscr{K} = O/P$ consists of a finite number q of elements, where q is a power of a prime number.

3. P is a principal ideal, that is, P contains an element $\mathfrak{p}$ such that $P = \mathfrak{p}O$. The norm of $\mathfrak{p}$ is

$$|\mathfrak{p}| = q^{-1},$$

where q is the order of the residue class field O/P.

Here are some examples:

1. $\mathbf{K}$ is the field of p-adic numbers. Here O consists of the elements of the form $\sum_{i=0}^{\infty} a_i p^i$ and its prime ideal P of the elements of the form $\sum_{i=1}^{\infty} a_i p^i$. Obviously P is generated by the number p, and $|p| = p^{-1}$.

2. $\mathbf{K}$ is the field of power series over the residue class field modulo p. Here O consists of the elements of the form $\sum_{i=0}^{\infty} a_i t^i$ and its prime ideal P of the elements of the form $\sum_{i=1}^{\infty} a_i t^i$. Obviously P is generated by the element t, and $|t| = p^{-1}$.

3. The multiplicative group of **K** contains an element ε of finite order $q - 1$ (where q is the order of the residue class field O/P). Clearly, $|\varepsilon| = 1$, that is, ε belongs to O but not to P. The elements $0, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{q-1} = 1$ form a complete set of representatives of the residue classes of O/P.

4. Every element of **K** has a unique representation as a convergent series

$$x = \mathfrak{p}^n(a_0 + a_1\mathfrak{p} + a_2\mathfrak{p}^2 + \cdots), \qquad a_0 \neq 0, \tag{1}$$

where $\mathfrak{p}$ is a generating element† of P, n an integer, and the coefficient a_i may assume the values $0, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{q-1} = 1$.

4. Additive and Multiplicative Characters of K. As an algebraic object the field **K** functions on two planes: it is a group under addition, and at the same time the set of elements of **K** other than 0 forms a group under multiplication. Henceforth we denote by $\mathbf{K}^+$ the additive group of **K**, and by $\mathbf{K}^*$ its multiplicative group. The most important functions on **K** are additive and multiplicative characters of **K**. Later we shall see that on the basis of these functions we can construct the theory of group representations, and in particular, the theory of special functions.

An additive character of **K** is a character of $\mathbf{K}^+$, that is, a continuous complex-valued function $\chi(x)$ satisfying the conditions:

1. $\chi(x + y) = \chi(x)\chi(y)$ for arbitrary elements x and y from K.
2. $|\chi(x)| = 1$.

A multiplicative character of **K** is a character of its multiplicative group $\mathbf{K}^*$, that is, a continuous complex-valued function $\pi(x)$ on **K** satisfying the conditions:

1. $\pi(xy) = \pi(x)\pi(y)$ for arbitrary elements x and y from $\mathbf{K}^*$.
2. $|\pi(x)| = 1$.

The additive and multiplicative characters themselves constitute topological groups, which we shall now describe.

The group of additive characters of a locally compact topological field‡ **K** *is isomorphic to its additive group* $\mathbf{K}^+$. This isomorphism is realized as follows. Let $\chi(x) \not\equiv 1$ be a fixed nontrivial additive character. Then it can be shown that every character on $\mathbf{K}^+$ is of the form

$$\chi_u(x) = \chi(ux),$$

† Thus, $\mathfrak{p} = p$ for the field Q_p of p-adic numbers; for the field $\mathbf{K}_q(t)$ of power series over a finite field we have $\mathfrak{p} = t$ (see the examples above).

We emphasize that for Q_p the representation (1) is not equivalent to the usual representation of a p-adic number in the form of a series (see **§ 1.1**). There the coefficients a_i were integers, $0 \leq a_i < p$; here, they are p-adic integers such that either $a_i^{p-1} = 1$, or $a_i = 0$.

‡ If **K** is a discrete field, then the group of additive characters is compact, and therefore not isomorphic to $\mathbf{K}^+$.

where u is an element of $\mathbf{K}$. The correspondence $u \to \chi_u(x)$ gives the required isomorphism of $\mathbf{K}^+$ with its character group.

We mention that for the field Q_p of p-adic numbers every character $\chi(ux)$ can be written in explicit form

$$\chi(ux) = e^{2\pi iux}.$$

The expression $e^{2\pi iux}$ has the following meaning. Since $e^{2\pi in} = 1$ for every integer n, the integral part of the p-adic number ux in the exponent on the right can be ignored. However, for extensions of Q_p such an expression for the characters is not available.

Now we proceed to a description of the multiplicative group $\mathbf{K}^*$ of $\mathbf{K}$ and its character group.

In accordance with **§ 1.3** (Proposition 4) we write every element of the field in the form

$$x = \mathfrak{p}^n \varepsilon^k (1 + a_1 \mathfrak{p} + a_2 \mathfrak{p}^2 + \cdots), \tag{1}$$

where $\mathfrak{p}$ is a generating element of the prime ideal P (in the ring of integers O), and the a_i take the values 0 or ε^l ($\varepsilon^{q-1} = 1$). The elements $\mathfrak{p}^n$ form an infinite cyclic subgroup of $\mathbf{K}^*$, and the elements ε^k a finite subgroup of order $q - 1$. It is clear that the elements $1 + a_1\mathfrak{p} + a_2\mathfrak{p}^2 + \cdots$ also form a subgroup of $\mathbf{K}^*$, and this is compact. Note that this subgroup can be described succinctly in terms of the norm: its elements are precisely those elements of $\mathbf{K}$ for which $|x - 1| < 1$.

Thus, the multiplicative group $\mathbf{K}^*$ *of* $\mathbf{K}$ *is a direct product*

$$\mathbf{K}^* = Z \times Z_{q-1} \cdot A$$

of three groups: the infinite cyclic group Z *of the elements* $\mathfrak{p}^n$, *the finite cyclic group* Z_{q-1} *of order* $q - 1$ *of the elements* ε^k, *and the compact group* A *of the elements* x *for which* $|x - 1| < 1$.

From this we can deduce the structure of the group multiplicative characters of $\mathbf{K}$. *The group of multiplicative characters of* $\mathbf{K}$ *is a direct product of three groups: the group of rotations of a circle, a cyclic group of order* $q - 1$, *and a certain infinite discrete group* (*the group dual to* A). Thus, every multiplicative character $\pi(x)$ is given by three quantities: a real number ρ, which is determined modulo 1; an integer α, which is determined modulo $q - 1$; and a character $\theta(a)$ of A. It is expressed by the following formula: if

$$x = \mathfrak{p}^n \varepsilon^k a, \tag{2}$$

where a belongs to A, then

$$\pi(x) = e^{2\pi in\rho} e^{2\pi i\,\alpha k/(q-1)} \theta(a). \tag{3}$$

In what follows we also consider nonunitary characters $\pi(x)$,

that is, continuous functions satisfying only the condition

$$\pi(xy) = \pi(x)\pi(y).$$

It is easy to verify that every such character $\pi(x)$, as before, is given by (3), in which ρ may now be an arbitrary complex number.

5. The Structure of the Subgroup A. **The functions** $\exp x$ **and** $\ln x$. Here we consider a disconnected field $\mathbf{K}$ of characteristic 0. Our aim is to study in detail the structure of the multiplicative group A of the elements x for which $|x - 1| < 1$. We show that under certain additional restrictions on $\mathbf{K}$ this subgroup is isomorphic to the additive group P of elements x for which $|x| < 1$.

The isomorphism $A \cong P$ is established by means of the functions $\exp x$ and $\ln x$, which we define via the sums of the power series:

$$\exp x = 1 + x + \frac{x^2}{2!} + \dots,$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \dots.$$

First we find out for what x these series are convergent.

Note that $\mathbf{K}$ is a finite extension of the field Q_p of p-adic numbers. The prime p is uniquely determined by $\mathbf{K}$: it is the only prime number for which $|p| < 1$; the norms of all other prime numbers are equal to 1.

The series for $\exp x$ *converges if and only if* $|x| < |p|^{1/(p-1)}$.

To prove this we begin by estimating $|n!|$. Let $p^k \leq n < p^{k+1}$. Then it is easy to verify that the power to which p occurs as a factor in $n!$ is†

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \dots + \left[\frac{n}{p^k}\right].$$

Consequently,

$$|n!| = |p|^{[n/p] + \dots + [n/p^k]},$$

and therefore,

$$|n|! \geq |p|^{n(1-p^{-k})/(p-1)}. \tag{1}$$

Suppose now that $|x| < |p|^{1/(p-1)}$ that is, $|x| = |p|^{(1+\varepsilon)/(p-1)}$, where $\varepsilon > 0$. Then we find on the basis of the estimate for $|n!|$

$$\left|\frac{x^n}{n!}\right| \leq |p|^{n(\varepsilon + p^{-k})/(p-1)}. \tag{2}$$

The convergence of the series $\exp x$ for $|x| < |p|^{1/(p-1)}$ follows immediately from this.

On the other hand, let $|x| \geq |p|^{1/(p-1)}$. Then for $n = p^k$ we have

$$|n!| = |p|^{n(1-p^{-k})/(p-1)},$$

therefore,

$$\left|\frac{x^n}{n!}\right| \geq |p|^{p^{-k}/(p-1)}.$$

From this estimate it is clear that for $|x| \geq |p|^{1/(p-1)}$ the series $\exp x$ diverges.

Let x belong to the domain of convergence of $\exp x$, that is, $|x| < |p|^{1/(p-1)}$.

† The symbol $[a]$ denotes the integral part of a.

Then

$$|\exp x - 1 - x| < |x|. \tag{3}$$

For from (2) we have, since $n \geq p^k$,

$$\left|\frac{x^n}{n!}\right| \leq |x|\,|p|^{(n-1)\varepsilon/(p-1)}$$

Consequently, $\left|\frac{x^n}{n!}\right| < |x|$ for $n \geq 2$. Hence, (3) follows immediately.

The series for $\ln(1 + x)$ *converges if and only if* $|x| < 1$.

If $|1 - y| < |p|^{1/(p-1)}$, *then* $|\ln y| = |1 - y|$.

We leave the verification of these statements to the reader.

It is easy to show that the functions $\exp x$ and $\ln x$ have the usual properties: $\exp(x_1 + x_2) = \exp x_1 \exp x_2$ $\ln(x_1 x_2) = \ln x_1 + \ln x_2$ (provided that the elements x_1 and x_2 lie in the domain of definition of the corresponding function).

The function $y = \exp x$ *effects an isomorphic map of the additive group* B *of elements for which* $|x| < |p|^{1/(p-1)}$ *onto the multiplicative group* A_1 *of elements* y *for which* $|1 - y| < |p|^{1/(p-1)}$. *The inverse isomorphism is given by the function* $x = \ln y$.

For let $x \in B$. Then the series $y = \exp x$ converges for x. From (3) it follows that $|\exp x - 1| = |x|$ so that $|1 - y| < 1$. But then the series $\ln y = \ln(\exp x)$ converges. By formal operations on series we verify that

$$\ln(\exp x) = x \tag{4}$$

for every x of B.

Conversely, let $y \in A_1$. Then the series $x = \ln y$ converges, and $|\ln y| = |1 - y|$; consequently, the series $\exp x = \exp(\ln y)$ also converges. By formal operations on series we verify that

$$\exp(\ln y) = y$$

for every y in A_1.

From (4) and (5) it follows that the function $y = \exp x$ effects a one-to-one map of B onto A_1. The fact that this is an isomorphism follows from the relation $\exp(x_1 + x_2) = \exp x_1 \exp x_2$.

Now let us find out under what conditions the subgroup A_1 coincides with the multiplicative group A of all elements x of the field for which $|1 - x| < 1$.

Let O be the ring of integers of $\mathbf{K}$, P the maximal ideal in O, $\mathfrak{p}$ a generating element of P, and $q^{-1} = |\mathfrak{p}|$ its norm.

Clearly, A consists of precisely those elements x for which $|1 - x| \leq q^{-1}$, where the equality sign may hold.

Consequently, the condition that $A = A_1$ can be expressed in the form

$$q^{-1} < |p|^{\frac{1}{p-1}}. \tag{6}$$

Let $|p| = q^{-(s-1)}$. This means that p belongs to P^{s-1}, but not to P^s. Then (6) can be rewritten in the form $q^{-1} < q^{\frac{s-1}{p-1}}$. From this we obtain the condition on s: $s < p$.

We state the final result. *Let* $\mathbf{K}$ *be a disconnected field of characteristic zero,* O *the subring of integers of* $\mathbf{K}$, P *the maximal ideal in* O, p *the characteristic of the residue class field* O/P. *We assume that* p *does not belong to* P^{p-1}. *Then the multiplicative group* A *of elements* x *of the field for which* $|1 - x| < 1$ *is isomorphic to the additive group* P *of elements* y *for which* $|y| < 1$. *The isomorphism is effected by the function* $y = \ln x$.

In general this statement is not true: A may contain elements of finite order p^n; then it is not isomorphic to any of the subgroups of the additive group P (because all elements of P are of infinite order).

6. Quadratic Extensions of a Disconnected Field. Let τ be an element of $\mathbf{K}$ that is not a square of another element of the field. By adjoining to $\mathbf{K}$ the square root $\sqrt{\tau}$, we obtain a quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$. The elements of $\mathbf{K}(\sqrt{\tau})$ have the form $z = x + \sqrt{\tau}\,y$, where x and y belong to $\mathbf{K}$. Addition and multiplication of such elements proceed in the usual way. Let us find out how many distinct quadratic extensions a disconnected field $\mathbf{K}$ has.

Obviously, two quadratic extensions $\mathbf{K}(\sqrt{x})$ and $\mathbf{K}(\sqrt{y})$ of $\mathbf{K}$ coincide if and only if the quotient xy^{-1} is a square in $\mathbf{K}$. In other words, there are as many quadratic extensions of $\mathbf{K}$ as there are nontrivial cosets of the multiplicative group $\mathbf{K}^*$ with respect to the subgroup of all squares $(\mathbf{K}^*)^2$.

We determine the index $\mathbf{K}^*:(\mathbf{K}^*)^2$. From **§ 1.4** we know that $\mathbf{K}^*$ is a direct product

$$\mathbf{K}^* = Z \times Z_{q-1} \times A$$

of an infinite cyclic group Z, a finite cyclic group Z_{q-1} of order $q - 1$, and the subgroup A of the element x in K for which $|x - 1| < 1$. Therefore,

$$\mathbf{K}^*:(\mathbf{K}^*)^2 = (Z:Z^2) \times (Z_{q-1}:Z_{q-1}^2) \times (A:A^2),$$

where Z^2, Z_{q-1}^2, A^2 are the subgroups consisting of the squares of the elements of the corresponding groups.

We now assume that q is an odd number, so that the residue class field O/P is of characteristic $p \neq 2$. In this case Z_{q-1} is a cyclic group of even order, and $Z_{q-1}:Z_{q-1}^2 = 2$. Also $Z:Z^2 = 2$.

We show, finally, that $A = A^2$, that is, that the equation $x^2 = a$ has a solution in A for every $a \in A$. For let

$$a = 1 + a_1\mathfrak{p} + a_2\mathfrak{p}^2 + \dots .$$

We are looking for a solution x of the equation $x^2 = a$ in the form of a series $x = 1 + x_1\mathfrak{p} + x_2\mathfrak{p}^2 + \dots$, where $x_i = 0$ or $x_i = \varepsilon^n$ (ε an element of O^* of order $q - 1$). The equation $x^2 = a$ reduces to a system of congruences

$$2x_1 \equiv a_1 (\text{mod } P), \qquad 2x_1 + (2x_2 + x_1^2)\mathfrak{p} \equiv a_1 + a_2\mathfrak{p} \ (\text{mod } P^2)$$

and so forth. Clearly, under the assumption made on q, $x_1, x_2, \dots$ can be found successively from these congruences.

So we obtain: *if the characteristic of the residue class field O/P is different from 2, then the squares of the elements $x \neq 0$ of $\mathbf{K}$ form a subgroup of index 4 of the multiplicative group of $\mathbf{K}$. There are then three distinct quadratic extensions of $\mathbf{K}$. Clearly, these quadratic extensions are $\mathbf{K}(\sqrt{\mathfrak{p}})$, $\mathbf{K}(\sqrt{\varepsilon\mathfrak{p}})$, and $\mathbf{K}(\sqrt{\varepsilon})$.*†

† Observe that the cases $\tau = \varepsilon\mathfrak{p}$ and $\tau = \mathfrak{p}$ are not distinct, because the element $\varepsilon\mathfrak{p}$ may play the role of p.

This result is not true when the characteristic of O/P is 2. For example, if $\mathbf{K}$ is of characteristic 2, then $A:A^2 = \infty$.

7. The Multiplicative Characters $\operatorname{sign}_\tau x$. Let $\mathbf{K}$ be a locally compact disconnected field. We assume, as before, that the finite residue class field O/P associated with it is not of characteristic 2. In this subsection we associate with every quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$ a certain multiplicative character $\operatorname{sign}_\tau x$ assuming the values ± 1 on $\mathbf{K}$.

Suppose then that $\mathbf{K}(\sqrt{\tau})$ is a quadratic extension of $\mathbf{K}$. We consider the product

$$z\bar{z} = x^2 - \tau y^2$$

of the element $z = x + \sqrt{\tau}\, y$ from $\mathbf{K}(\sqrt{\tau})$ by its conjugate† $\bar{z} = x - \sqrt{\tau}\, y$.

The set of elements $z\bar{z}$, $z \neq 0$, forms a subgroup $\mathbf{K}^*_\tau$ of the multiplicative group $\mathbf{K}^*$; and $\mathbf{K}^*_\tau$ obviously contains $(\mathbf{K}^*)^2$.

Let us show that *the index* $\mathbf{K}^*:\mathbf{K}^*_\tau$ *is* 2.

It is enough to verify that $\mathbf{K}^*_\tau \neq \mathbf{K}^*$ and $\mathbf{K}^*_\tau \neq (\mathbf{K}^*)^2$. Our assertion then follows immediately from the fact that $\mathbf{K}^*:(\mathbf{K}^*)^2 = 4$ (see **§ 1.6**).

We begin by showing that $\mathbf{K}^*_\tau \neq (\mathbf{K}^*)^2$. For if $\tau = \mathfrak{p}$ or $\tau = \varepsilon\mathfrak{p}$, then $-\tau$ is not a square of an element from $\mathbf{K}^*$, but belongs to $\mathbf{K}^*_\tau$. Now let $\tau = \varepsilon$. It can be shown that there exists integers x and y such that $x^2 - \varepsilon y^2 \equiv \varepsilon (\operatorname{mod} P)$‡ It is obvious that then $x^2 - \varepsilon y^2$ is not a square, but belongs to $\mathbf{K}^*_\tau$. Hence $\mathbf{K}^*_\tau \neq (\mathbf{K}^*)^2$. Now we show that $\mathbf{K}^*_\tau \neq \mathbf{K}^*$. For in the case $\tau = \mathfrak{p}$ or $\tau = \varepsilon\mathfrak{p}$ the element ε does not belong to $\mathbf{K}^*_\tau$. But for $\tau = \varepsilon$ the subgroup $\mathbf{K}^*_\tau$ cannot contain $\mathfrak{p}$ (otherwise we have a congruence $x^2 - \varepsilon y^2 \equiv 0$ $(\operatorname{mod} P)$ for certain $x \not\equiv 0$ $(\operatorname{mod} P)$ and $y \not\equiv 0$ $(\operatorname{mod} P)$, which is impossible).

Hence $\mathbf{K}^*_\tau \neq (\mathbf{K}^*)^2$, $\mathbf{K}^*_\tau \neq \mathbf{K}^*$, and $\mathbf{K}^*:\mathbf{K}^*_\tau = 2$.

Now we introduce the function $\operatorname{sign}_\tau x$ on $\mathbf{K}^*$. We set

$$\operatorname{sign}_\tau x = 1,$$

when $x \in \mathbf{K}^*_\tau$, that is, when x is representable in the form $x = x_1^2 - \tau x_2^2$ and

$$\operatorname{sign}_\tau x = -1,$$

when x is not so representable.

† The expression $z\bar{z}$ is often called the norm of z relative to $\mathbf{K}$.

‡ This results from the following theorem. Let F be a finite field, and ε an element of the field that is not a square; then every element x of the field can be represented in the form $x = x_1^2 - \varepsilon x_2^2$, where $x_1, x_2 \in F$ (see [42]).

From the fact that $\mathbf{K}_\tau^*$ is a subgroup of index 2 in $\mathbf{K}^*$ it follows immediately that $\operatorname{sign}_\tau x$ *is a character on* $\mathbf{K}^*$, *that is,*

$$\operatorname{sign}_\tau (xy) = \operatorname{sign}_\tau x \cdot \operatorname{sign}_\tau y$$

for arbitrary x and y of $\mathbf{K}^*$.

We call elements x of $\mathbf{K}$ positive or negative (it would be more accurate to say τ-positive or τ-negative) according to the sign of $\operatorname{sign}_\tau x$.

It can be shown that the functions $\operatorname{sign}_\tau x$, where $\tau = \mathfrak{p}, \varepsilon\mathfrak{p}, \varepsilon$, are independent. Therefore, together with $\pi_0(x) \equiv 1$ they form a complete system of characters on the factor group $\mathbf{K}^*/(\mathbf{K}^*)^2$.

8. Circles in $\mathbf{K}(\sqrt{\tau})$. Let $\mathbf{K}(\sqrt{\tau})$ be a quadratic extension of a disconnected field $\mathbf{K}$. The set of elements z of $\mathbf{K}(\sqrt{\tau})$ that satisfy the equation

$$z\bar{z} = c, \qquad c \neq 0,$$

is called a *circle* in $\mathbf{K}(\sqrt{\tau})$ (with center at 0).

Observe that in contrast to the field of real numbers there are two types of circles: circles of "real" radius for which c is a square of an element of $\mathbf{K}$, and circles of "imaginary" radius for which c is not a square.

A special role is played by the circle

$$z\bar{z} \equiv x^2 - \tau y^2 = 1$$

whose elements form a group under multiplication and which we denote by C_τ.

We now give a parametric equation for the circle

$$x^2 - \tau y^2 = 1.$$

We use the parameter $\dfrac{y}{x+1} = t$. From the equation of the circle it follows that

$$\frac{x-1}{x+1} = \tau \frac{y^2}{(x+1)^2} = \tau t^2,$$

hence,

$$x = \frac{1+\tau t^2}{1-\tau t^2} \qquad y = (x+1)t = \frac{2t}{1-\tau t^2}.$$

Thus, *the circle $x^2 - \tau y^2 = 1$ is given by the following parametric representation;*

$$x = \frac{1+\tau t^2}{1-\tau t^2} \qquad y = \frac{2t}{1-\tau t^2}, \tag{1}$$

Next we show that *all circles are compact.*

It is sufficient to consider the unit circle, because every other circle consists of the points $w = az$, where z ranges over $z\bar{z} = 1$. Obviously the set of points of the circle $z\bar{z} = 1$ is closed. On the other hand, from the parametric equations (1) it follows that $|x| \le 1$, $|y| \le 1$; consequently, the set of points of the circle lies in a bounded domain and is therefore compact.

Let us study the detailed structure of the group C_τ of elements z, $z\bar{z} = x^2 - \tau y^2 = 1$. To begin with, let $\tau = \mathfrak{p}$ or $\tau = \varepsilon\mathfrak{p}$. In this case we have $|x^2| = 1$, $|\tau y^2| < 1$. Consequently, $|1 - x^2| < 1$, and therefore either $|1 - x| < 1$ or $|1 + x| < 1$. Hence we conclude: for $\tau = \mathfrak{p}$ or $\tau = \varepsilon\mathfrak{p}$ the group C_τ is the direct product

$$C_\tau = Z_2 \times C_\tau'$$

of a cyclic group of order 2, $Z_2 = \{1, -1\}$, and the subgroup C_τ' of elements of C_τ for which $|z - 1| < 1$.†

Now we take the case $\tau = \varepsilon$. The elements of the circle $z\bar{z} = 1$ may be written in the form of a series

$$z = (a_0 + \sqrt{\varepsilon}b_0)[1 + (a_1 + \sqrt{\varepsilon}b_1)\mathfrak{p} + (a_2 + \sqrt{\varepsilon}b_2)\mathfrak{p}^2 + \cdots],$$

where a_i and b_i take the values 0 and ε^k, $k = 0, 1, \ldots, q - 1$. From $z\bar{z} = 1$ it follows that

$$a_0^2 - \varepsilon b_0^2 \equiv 1 \pmod{P}.$$

It can be shown that this congruence has $q + 1$ solutions, where q is the order of the residue class field O/P.‡

We conclude: *let C_ε' be the subgroup of C_ε that consists of the elements z for which* $|z - 1| < 1$; *then the index of C_ε' in C_ε is $q + 1$.*

9. Cartesian and Polar Coordinates in $\mathbf{K}(\sqrt{\tau})$. Every element of $\mathbf{K}(\sqrt{\tau})$ has a unique representation in the form

$$z = x + \sqrt{\tau}y,$$

where $x, y \in \mathbf{K}$. Hence, it is given by a pair of elements x and y from $\mathbf{K}$, which we call the Cartesian coordinates of z.

Now we introduce polar coordinates of z. Let $z\bar{z} = c$. Then two cases are possible: either c is a square of an element of $\mathbf{K}$, or it is not a square.

First, let $c = r^2$, $r \in \mathbf{K}$. Then we define the polar coordinates of z as a pair: the element $\rho = r \in \mathbf{K}$ and the element $t = \rho^{-1}z$, which belongs to the circle $t\bar{t} = 1$. Clearly the point z is uniquely determined by its polar coordinates.

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† We define the norm $|z|$ on $\mathbf{K}(\sqrt{\tau})$ relative to the norm $|x|$ on $\mathbf{K}$ by the following formula: $|z| = |z\bar{z}|^{1/2}$.

‡ This follows from a theorem on finite fields: the equation $x^2 - \omega y^2 = 1$, where ω is not a square has $q + 1$ solutions in the finite field of order q. (The theorem then follows immediately from the parametric equations of the circle.)

Observe that ρ is determined to within its sign. Consequently, $(-\rho, -t)$ can equally well be regarded as the polar coordinates of z. Thus, the polar coordinates of z are determined to within a sign.

Now we take the case when c is not a square. In $\mathbf{K}(\sqrt{\tau})$ we fix an arbitrary element ν such that $\nu\bar{\nu}$ is not a square in $\mathbf{K}$. Then c can be represented in the form $c = (\nu r)(\bar{\nu} r)$, where $r \in \mathbf{K}$. We now define the polar coordinates of z as the pair of elements $\rho = \nu r$ and $t = \nu^{-1}z$, of which the latter is again a point of the unit circle. As in the first case, we have

$$(\rho, t) = (-\rho, -t).$$

10. Invariant Measures on K and Its Quadratic Extension $\mathbf{K}(\sqrt{\tau})$. There are two invariant measures on $\mathbf{K}$: a measure dx, invariant under addition ($d(x + a) = dx$), and a measure d^*x, invariant under multiplication ($d^*(xa) = d^*x$). These measures are very simply connected:

$$d^*x = |x|^{-1}\, dx. \tag{1}$$

For by definition of $|x|$, we have $d(xa) = |a|\, dx$. Consequently, $|xa|^{-1}\, d(xa) = |x|^{-1}\, dx$, so that the measure $|x|^{-1}\, dx$ is invariant under multiplication.

We always normalize dx by the following condition:

$$\int\limits_{|x| \leq 1} dx = 1.$$

We now consider the measures dz and d^*z, $z = x + \sqrt{\tau}y$, on $\mathbf{K}(\sqrt{\tau})$ that are invariant under addition and multiplication, respectively. Expressing these measures in terms of the Cartesian coordinates x and y of z we find

$$dz = dx\, dy, \qquad d^*z = \frac{dx\, dy}{|x^2 - \tau y^2|}.$$

Next we express the measures dz and d^*z in terms of the polar coordinates (ρ, t) of z. We recall that the coordinate ρ is determined to within its sign from the equation $\rho\bar{\rho} = z\bar{z}$, and that the second coordinate $t = \rho^{-1}z$ is a point on the circle $t\bar{t} = 1$.

Since the circle $t\bar{t} = 1$ is a group under multiplication, there is an invariant measure d^*t on it, and we normalize the measure by the condition

$$\int\limits_{t\bar{t}=1} d^*t = 1.$$

It is easy to check that in polar coordinates the measures dz

and d^*z are expressed by the following formulae:

$$dz = a_\tau\, d(zz)\, \bar{d}^*t, \qquad d^*z = a_\tau \frac{d(z\bar{z})\, d^*t}{|z\bar{z}|},$$

where $d(z\bar{z})$ is the measure on $\mathbf{K}$ and $a_\tau = 2(1 + q^{-1})(1 + |\tau|)^{-1}$.

11. Additive and Multiplicative Characters on the "Plane" $\mathbf{K}(\sqrt{\tau})$. The additive group of $\mathbf{K}(\sqrt{\tau})$ is the direct sum of two groups isomorphic to the additive group of $\mathbf{K}$. From this it follows that every additive character $\chi(z)$, $z = x + \sqrt{\tau}\, y$, of $\mathbf{K}(\sqrt{\tau})$ is of the form

$$\chi(z) = \chi_1(x)\chi_2(y), \tag{1}$$

where χ_1 and χ_2 are additive characters on $\mathbf{K}$.

Now we proceed to a description of the multiplicative characters on $\mathbf{K}(\sqrt{\tau})$.

For this purpose we study in detail the multiplicative group of $\mathbf{K}(\sqrt{\tau})$. In accordance with § **1.8** we represent every element of the field in the form $z = rt$ or $z = \nu rt$, where $r \in \mathbf{K}$, $t\bar{t} = 1$, and ν is a fixed element from $\mathbf{K}(\sqrt{\tau})$ for which $\nu\bar{\nu}$ is not a square of an element from $\mathbf{K}$.

Let $\pi(z)$ be a multiplicative character on $\mathbf{K}(\sqrt{\tau})$. We denote by π_1 and π_2 its restrictions to $\mathbf{K}$ and to the circle $t\bar{t} = 1$, respectively. Then we have

$$\pi(rt) = \pi_1(r)\pi_2(t). \tag{2}$$

From the equation $rt = (-r)(-t)$ we obtain a condition connecting π_1 and π_2:

$$\pi_1(-1) = \pi_2(-1). \tag{3}$$

Furthermore, since $\nu\bar{\nu} = r_0 \in K$, we have $\nu^2 = r_0 t_0$, where $t_0\bar{t}_0 = 1$. Consequently, $\pi(\nu^2) = \pi_1(r_0)\pi_2(t_0)$, that is,

$$\pi^2(\nu) = \pi_1(\nu\bar{\nu})\pi_2\left(\frac{\nu}{\bar{\nu}}\right). \tag{4}$$

Suppose, conversely, that π_1 and π_2 are arbitrary multiplicative characters on $\mathbf{K}$ and on $t\bar{t} = 1$, respectively, and connected by (3). We define $\pi(\nu)$ so that (4) is satisfied and then give the function $\pi(z)$ on $\mathbf{K}(\sqrt{\tau})$ by the following formulae:

$$\pi(rt) = \pi_1(r)\pi_2(t), \tag{5}$$

$$\pi(\nu rt) = \pi(\nu)\pi(rt). \tag{6}$$

Obviously, this function is a multiplicative character on $\mathbf{K}(\sqrt{\tau})$. Thus, *a multiplicative character of* $\mathbf{K}(\sqrt{\tau})$ *is given by its values on the*

ground field **K** *and on the circle* $t\bar{t} = 1$ *and its value at a fixed point* ν *such that* $\nu\bar{\nu}$ *is not a square in* **K**.† These values are linked by the relations (3) and (4).

§ 2. TEST AND GENERALIZED FUNCTIONS ON A LOCALLY COMPACT DISCONNECTED FIELD K

In this section we discuss certain problems of analysis on a locally compact disconnected topological field **K**.

1. The Space of Test Functions. Let **K** be a locally compact disconnected field. We recall that **K** contains a decreasing sequence of subrings

$$P \supset P^2 \supset \cdots \supset P^n \supset \cdots$$

(where P is the maximal ideal of the ring of integers in **K**) that are open compact sets and form a complete system of neighborhoods of zero.

We wish to lay down a set of sufficiently well-behaved functions on **K**. For this purpose we consider the set S of all complex-valued functions $f(x)$ on **K** that satisfy the following two requirements:

1. The function $f(x)$ is finite, that is, equal to 0 outside some compact open set.

2. There exists a positive integer n (depending on $f(x)$) such that $f(x)$ is constant on the cosets $\mathbf{K}/P^n$.

From 2. it follows automatically that $f(x)$ is a continuous function on **K**. Clearly, the set S of these functions $f(x)$ forms a linear space. We now introduce a topology in S.

We say that a sequence of functions $f_i(x)$ tends to zero if:

1. The functions $f_i(x)$ are zero outside some fixed compact set (independent of i).

2. There exists a positive integer n such that all the functions $f_i(x)$ are constant on the cosets $\mathbf{K}/P^n$.

3. The sequence $f_i(x)$ tends to zero uniformly in x, as $i \to \infty$.

It is easy to verify that *with this topology* S *becomes a complete linear space*, which we call the space of *test* functions. A *generalized function* $\varphi(x)$ is a continuous functional (φ, f) on S.

By analogy with S we can also define the space S_n of functions $f(x_1, \ldots, x_n)$ of n variables from **K**.

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† Observe that the value of the character π at the point ν is determined to within its sign, in accordance with (4), by its values on **K** and on the circle $t\bar{t} = 1$.

2. Generalized Functions Concentrated at a Point. As usual, we define the generalized function $\delta(x)$ by the following formula:

$$(\delta(x), f(x)) = f(0).$$

It is not hard to see that *every generalized function concentrated at* $x = 0$ *is, to within a factor, the function* $\delta(x)$.

This follows immediately from the fact that every test function $f(x) \in S$ is constant in a neighborhood of $x = 0$.

Of course, this statement is also true for generalized functions of n variables.

3. Homogeneous Generalized Functions. Let $\pi(x)$ be a multiplicative character on $\mathbf{K}$, that is,

$$\pi(xy) = \pi(x)\pi(y)$$

for arbitrary x and y from $\mathbf{K}$ (we do not require that $|\pi(x)| = 1$). We call a generalized function $\varphi(x)$ *homogeneous of degree* π if for every function $f \in S$ and $t \neq 0$ we have

$$(\varphi, f(t^{-1}x)) = \pi(t)\ |t|\ (\varphi, f(x)). \tag{1}$$

Our task is to describe all the homogeneous generalized functions. According to **§ 1.4** the multiplicative character $\pi(x)$ can be represented in the form

$$\pi(x) = |x|^{s-1}\, \theta(x), \tag{2}$$

where s is a complex number, and $\theta(x)$ is another character on $\mathbf{K}$ such that

$$|\theta(x)| = 1, \tag{3}$$

$$\theta(\mathfrak{p}) = 1. \tag{4}$$

By (4), θ is given by its values on the compact subgroup of elements x of norm $|x| = 1$. Consequently, the set of these characters θ is discrete.

With the character $\pi(x)$ we associate a generalized function $\pi(x)$ defined by the formula

$$(\pi(x), f(x)) = \int \pi(x) f(x)\, dx \equiv \int |x|^{s-1}\, \theta(x) f(x)\, dx. \tag{5}$$

For $\operatorname{Re} s > 0$ this integral converges in the usual sense and is an analytic function of s. For $\operatorname{Re} s < 0$ we define it by means of analytic continuation.

It is not difficult to see that $\pi(x)$ *is a homogeneous generalized function of degree* π, *provided* s *is a nonsingular point of the integral* (5).

We show now that *the only singularity of the generalized function* $\pi(x) = |x|^{s-1}\theta(x)$, *regarded as a function of the discrete argument* θ *and*

as an analytic function of s, is the point $\theta \equiv 1$, $s = 0$. At this point $\pi(x)$, as a function of s, has a simple pole with the residue $\frac{q-1}{q \ln q}\,\delta(x)$.

Proof. Without loss of generality we may assume that $f(x)$ is concentrated in the domain $|x| \leq 1$. We rewrite the expression (5) in the form

$$(\pi, f) = \int_{|x| \leq 1} |x|^{s-1}\theta(x)[f(x) - f(0)]\, dx + f(0) \int_{|x| \leq 1} |x|^{s-1}\, \theta(x)\, dx.$$

The first integral converges for arbitrary s, because the function $f(x) - f(0)$ is equal to 0 in the neighborhood of $x = 0$. Therefore we examine the second integral. We split it into the sum of the integrals

$$\int_{|x| \leq 1} |x|^{s-1}\, \theta(x)\, dx = \sum_{k=0}^{\infty} q^{-k(s-1)} \int_{|x| = q^{-k}} \theta(x)\, dx.$$

If $\theta(x) \not\equiv 1$, then $\int_{|x| = q^{-k}} \theta(x)\, dx = 0$ for every k. Thus we are left with the case $\theta(x) \equiv 1$, that is, we are led to compute the integral

$$\int_{|x| \leq 1} |x|^{s-1}\, dx = \sum_{k=0}^{\infty} q^{-k(s-1)} \int_{|x| = q^{-k}} dx.$$

We recall that the measure dx is normalized so that

$$\int_{|x| \leq 1} dx = 1.$$

It follows that

$$\int_{|x| \leq q^{-k}} dx = \int_{|y| \leq 1} d(\mathrm{p}^k y) = q^{-k}.$$

Consequently,

$$\int_{|x| = q^{-k}} dx = \int_{|x| \leq q^{-k}} dx - \int_{|x| \leq q^{-k-1}} dx = q^{-k}(1 - q^{-1}).$$

Thus,

$$\int_{|x| \leq 1} |x|^{s-1}\, dx = (1 - q^{-1}) \sum_{k=0}^{\infty} q^{-ks} = \frac{1 - q^{-1}}{1 - q^{-s}}.$$

So we see that the only singularity of this expression is a simple pole at $s = 0$, and here

Thus

$$\operatorname*{Res}_{s=0} (|x|^{s-1}, f(x)) = \frac{q-1}{q \ln q} f(0),$$

that is,

$$\operatorname*{Res}_{s=0} |x|^{s-1} = \frac{q-1}{q \ln q} \delta(x).$$

This proves the assertion. We formulate the final result.

To every multiplicative character $\pi(x)$, except $\pi_0(x) = |x|^{-1}$ there corresponds a homogeneous generalized function $\pi(x)$ of degree π, defined by (5). *Obviously the function $\delta(x)$ is homogeneous of degree π_0.*

We show now that *other homogeneous generalized functions do not exist.*

Let $\varphi(x)$ be a homogeneous generalized function of degree π, $\pi(x) \neq |x|^{-1}$. It is not difficult to check that for functions $f(x)$ that are zero in a neighborhood of $x = 0$ we have

$$(\varphi, f) = c(\pi, f),$$

where c is some constant. Consequently, the function $\varphi(x) - c\pi(x)$ is concentrated at $x = 0$, and so $\varphi(x) - c\pi(x) = c_1 \delta(x)$. But the functions $\varphi(x) - c\pi(x)$ and $\delta(x)$ have different degrees of homogeneity. Consequently, $c_1 = 0$, that is, $\varphi(x) = c\pi(x)$.

Now let $\varphi(x)$ be a homogeneous generalized function of degree π_0, $\pi_0(x) = |x|^{-1}$. We show that $\varphi(x)$ is concentrated at $x = 0$ and therefore that $\varphi(x) = c\,\delta(x)$. For suppose that $\varphi(x)$ is not concentrated at $x = 0$. Then we can easily show that

$$(\varphi, f) = c\int |x|^{-1} f(x)\, dx, \qquad c \neq 0,$$

for every function f that is zero in a neighborhood of 0. We introduce the generalized function φ_1:

$$(\varphi_1, f) = c \int_{|x| \leq 1} |x|^{-1} [f(x) - f(0)]\, dx + c \int_{|x| > 1} |x|^{-1} f(x)\, dx. \quad (6)$$

For every function f that is equal to zero in a neighborhood of $x = 0$ we have $(\varphi, f) = (\varphi_1, f)$. Consequently, the function $\varphi - \varphi_1$ is supported at $x = 0$, and so $\varphi - \varphi_1 = c\,\delta(x)$. But the functions $\varphi(x)$ and $\delta(x)$ are homogeneous of one and the same degree π_0. Hence φ_1 must also be homogeneous. However, as is easily verified from (6), φ_1 is not homogeneous. This contradiction shows that $\delta(x)$ is the only homogeneous function of degree π_0, $\pi_0(x) = |x|^{-1}$.

4. The Fourier Transform of Test Functions. Let $\chi(x) \not\equiv 1$ be an additive character on $\mathbf{K}$. We define the Fourier transform of $f(x)$ by the formula

$$\tilde{f}(u) = \int \chi(ux) f(x)\, dx. \tag{1}$$

The Fourier transform is defined for every function $f(x)$ of integrable square modulus; the integral (1) must then be understood in the sense of the mean square value. It is known that $f(x)$ is expressed in terms of its Fourier transform by the formula

$$f(x) = c \int \chi(-ux) \tilde{f}(u)\, du. \tag{2}$$

where c is a positive constant dependent on the choice of the character χ. Moreover, the Plancherel formula holds:

$$\int |f(x)|^2\, dx = c \int |\tilde{f}(u)|^2\, du. \tag{3}$$

Let us find out how the constant c depends on the choice of the character χ. From the continuity of χ it follows that $\chi(x) \equiv 1$ on the subgroup $\mathfrak{p}^k O$ for sufficiently large k, where O is the subgroup of elements x of norm $|x| \leq 1$. We define the *rank* of the character χ as the least integer n such that $\chi(x) \equiv 1$ on $\mathfrak{p}^n O$. Clearly, if χ is of rank n, then the character $\chi'(x) = \chi(\mathfrak{p}^k x)$ is of rank $n - k$.

We show that the constant c in the inversion formula (2) and the Plancherel formula (3) can be expressed in terms of the rank of χ as follows:

$$c = q^n, \tag{4}$$

with $q^{-1} = |\mathfrak{p}|$. For this purpose we denote by ψ the characteristic function of O and compute its Fourier transform. We find

$$\tilde{\psi}(u) = \int_K \psi(x) \chi(ux)\, dx = \int_O \chi(ux)\, dx.$$

We represent the element u in the form $\mathfrak{p}^k v$, $|v| = 1$. Then

$$\tilde{\psi}(u) = \int_O \chi(\mathfrak{p}^k v x)\, dx = |\mathfrak{p}|^{-k} \int_{\mathfrak{p}^k O} \chi(y)\, dy. \tag{5}$$

Suppose that the rank of χ is n. For $k \geq n$ the function under the integral sign in (5) is equal to 1, and we obtain

$$\tilde{\psi}(u) = |\mathfrak{p}|^{-k} \int_{\mathfrak{p}^k O} dy = \int_O dx = 1.$$

But if $k < n$, then χ is a nontrivial character on $p^k O$; and therefore, the integral is equal to 0.

This result can be written as follows:

$$\tilde{\psi}(u) = \begin{cases} 1, & \text{when} \quad |u| \leq q^{-n}, \\ 0, & \text{when} \quad |u| > q^{-n}, \end{cases} \tag{6}$$

that is, $\tilde{\psi}$ is the characteristic function of $\mathfrak{p}^n O$.

Substituting ψ and $\tilde{\psi}$ in the Plancherel formula we obtain the required equation (4). In particular, if the rank of χ is zero, then $c = 1$.

Henceforth we always assume that the character χ in the definition of the Fourier transform is of rank 1 so that

$$c = 1.$$

First, we discuss the Fourier transforms of the test functions.

The Fourier transform of a function $f \in S$ is also a function in S.

Proof. Let $f(x)$ be a function in S. This means that:

1. There is an m such that $f(x) = 0$ for $|x| \geq q^m$.

2. There is an n such that for every t of norm $|t| \leq q^{-n}$ we have $f(x + t) = f(x)$.

Consider the Fourier transform of $f(x)$:

$$\tilde{f}(u) = \int \chi(ux) f(x)\, dx. \tag{7}$$

First let us show that $\tilde{f}(u)$ is a finite function. For this purpose we replace x by $x + t$ under the integral, where $|t| \leq q^{-n}$. By 2. we obtain

$$\tilde{f}(u) = \chi(ut) \int \chi(ux) f(x)\, dx,$$

that is,

$$\tilde{f}(u) = \chi(ut) \tilde{f}(u). \tag{8}$$

If $|u| > q^n$, then $|ut| > 1$ and hence $\chi(ut) \neq 1$. But then it follows from (8) that $\tilde{f}(u) = 0$ when $|u| > q^n$. This shows that $\tilde{f}(u)$ is a finite function.

Next, we show that $\tilde{f}(u)$ satisfies condition 2.

Since $\tilde{f}(x) = 0$ for $|x| \geq q^m$, we have

$$\tilde{f}(u) = \int_{|x| \leq q^m} \chi(ux) f(x)\, dx.$$

Consequently, for $|t| \leq q^{-m}$ we find

$$\tilde{f}(u + t) = \int_{|x| \leq q^m} \chi(tx) \chi(ux) f(x)\, dx = \tilde{f}(u),$$

because $\chi(tx) = 1$. Hence $\tilde{f}(u)$ satisfies condition 2, and the proposition is proved.

Note that $\tilde{\tilde{f}}(x) = f(-x)$. Hence it follows immediately that:

The Fourier transform effects a one-to-one map of the space S of test functions onto itself.

Now we give a definition of the Fourier transform of a generalized function. As a basis for this definition we use the Plancherel formula

$$\int \varphi(x)\overline{f(x)}\,dx = \int \tilde{\varphi}(u)\overline{\tilde{f}(u)}\,du, \tag{9}$$

which holds for arbitrary test functions f and φ. It is not difficult to see that the function $\tilde{f}(u)$ is the Fourier transform of $\overline{f(-x)}$. Thus, if in (9) we replace $f(x)$ by $\overline{f(-x)}$, we obtain

$$\int \varphi(x)f(-x)\,dx = \int \tilde{\varphi}(u)\tilde{f}(u)\,du. \tag{10}$$

The equation (1) means that the function $\tilde{\varphi}(u)$, regarded as a functional, satisfies the following relation:

$$(\tilde{\varphi}, \tilde{f}(u)) = (\varphi, f(-x)). \tag{11}$$

We take this relation as the definition of the Fourier transform of the generalized functions $\varphi(x)$. Thus, *the Fourier transform of the generalized function $\varphi(x)$ is the generalized function $\tilde{\varphi}(u)$ defined by* (11).

5. The Fourier Transform of Homogeneous Generalized Functions. The Gamma-Function and Beta-Function. From the definition of the Fourier transform we deduce immediately that

$$\tilde{1} = \delta(x), \qquad \widetilde{\delta(x)} = 1. \tag{1}$$

Now we show that *the Fourier transform of a homogeneous generalized function of degree π is homogeneous of degree $\pi^{-1}\pi_0^{-1}$, where $\pi_0(x) = |x|^{-1}$.*

For let φ be a homogeneous function of degree π. This means that for every $t \neq 0$ from K we have

$$(\varphi, f(t^{-1}x)) = \pi\pi_0(t)(\varphi, f(x)),$$

where

$$\pi_0(t) = |t|(\pi\pi_0(t) \equiv \pi(t)\pi_0(t)).$$

Now we observe that when $f_1(x) = |t|\,f(tx)$, then $\tilde{f}_1(u) = \tilde{f}(t^{-1}u)$. Consequently,

$$(\tilde{\varphi}, \tilde{f}(t^{-1}u)) = (\varphi, |t|\,f(-tx)) = \pi^{-1}(t)(\varphi, f(-x)),$$

that is,

$$(\tilde{\varphi}, \tilde{f}(t^{-1}u)) = \pi^{-1}(t)(\tilde{\varphi}, \tilde{f}(u)).$$

This equation means that $\tilde{\varphi}$ is a homogeneous function of degree $\pi^{-1}\pi_0^{-1}$.

Thus, the Fourier transform of the homogeneous generalized function $\pi(x)\ |x|^{-1}$ is, to within a factor, the homogeneous generalized function $\pi^{-1}(u)$. We denote the factor arising here by $\Gamma(\pi)$ and call it the *Gamma-function.* So we have

$$\widetilde{\pi(x)\ |x|^{-1}} = \Gamma(\pi)\pi^{-1}(u). \tag{2}$$

Let us find an integral representation of $\Gamma(\pi)$. For this purpose we write $\widetilde{\pi(x)\ |x|^{-1}}$ in the form of an integral

$$\Gamma(\pi)\pi^{-1}(u) = \int \chi(ux)\pi(x)\ |x|^{-1}\,dx.$$

By taking $u = 1$ we find

$$\Gamma(\pi) = \int \chi(x)\pi(x)\ |x|^{-1}\,dx. \tag{3}$$

Clearly, this expression is reminiscent of the formula for the classical Gamma-function.†

We can give a meaning to the integral (3) by writing it as the sum of the two integrals

$$\Gamma(\pi) = \int\limits_{|x|\leqslant 1} \chi(x)\pi(x)\ |x|^{-1}\,dx + \int\limits_{|x|>1} \chi(x)\pi(x)\ |x|^{-1}\,dx.$$

Each of these integrals converges in a certain domain of values of π and is in this domain an analytic function of π. By analytic continuation we define these integrals for arbitrary π.

By splitting the integrals (3) into a sum of integrals over domains $|x| = \text{const}$ we find, after a suitable change of variables, the following expression for the function $\Gamma(\pi)$ (*expansion in a Fourier series*):

$$\Gamma(\pi) = \sum_{k=-\infty}^{+\infty} \pi(\mathfrak{p}^k) \int\limits_{|x|=1} \chi(\mathfrak{p}^k x)\pi(x)\,dx. \tag{4}$$

† We mention that in the case of the field of real numbers the Gamma-function we have introduced does not coincide with the classical one, but differs from it by a factor. For example, if $\pi(x) = |x|^s$, then

$$\Gamma(\pi) = \int\limits_{-\infty}^{+\infty} |x|^{s-1} e^{ix}\,dx = 2\cos\frac{\pi s}{2}\,\Gamma(s).$$

where $\Gamma(s)$ is the classical Gamma-function. Similarly, for $\pi(x) = |x|^s \operatorname{sign} x$ we have $\Gamma(\pi) = 2i \sin\frac{\pi s}{2}\,\Gamma(s)$.

The following properties of the Gamma-function are immediate consequences of the definition:

1. The only singular point of $\Gamma(\pi)$ is $\pi \equiv 1$.
2. The only zero of $\Gamma(\pi)$ is $\pi_0(x) = |x|$.
3. $\Gamma(\pi)\,\Gamma(\pi_0\pi^{-1}) = \pi(-1)$. (5)

To obtain (5) we apply the Fourier transform to both sides of (2).

Note that the formula (5) is reminiscent of the relation (for the classical Gamma-function) connecting $\Gamma(t)$ and $\Gamma(1 - t)$. Now we give a definition of the Beta-function.

The Beta-function of the multiplicative characters π_1 and π_2 of $\mathbf{K}$ is the following expression:

$$B(\pi_1, \pi_2) = \int \pi_1(x)\ |x|^{-1}\ \pi_2(1 - x|^{-1}\,dx. \tag{6}$$

The integral diverges and must be understood in the following sense. We split (6) into two integrals:

$$B(\pi_1, \pi_2) = \int\limits_{|x|\leqslant 1} \pi_1(x)\ |x|^{-1}\ \pi_2(1 - x)\ |1 - x|^{-1}\,dx + \int\limits_{|x|>1} \pi_1(x)\ |x|^{-1}\ \pi_2(1 - x)\ |1 - x|^{-1}\,dx.$$

Each of these integrals converges in a certain domain of the characters π_1 and π_2 and is, in this domain, an analytic function of π_1 and π_2. By analytic continuation we define these integrals for all π_1 and π_2. In this way we define $B(\pi_1, \pi_2)$ as an analytic function of π_1 and π_2.

It is easy to show that the function $B(\pi_1, \pi_2)$ has the following expression in terms of the Gamma-function

$$B(\pi_1, \pi_2) = \frac{\Gamma(\pi_1)\,\Gamma(\pi_2)}{\Gamma(\pi_1\pi_2)}\,. \tag{7}$$

The derivation of this formula proceeds just as for the classical Beta- and Gamma-functions.

6. Additional Information on the Gamma-Function. We recall that the multiplicative group $\mathbf{K}^*$ of a disconnected topological field $\mathbf{K}$ is the direct product of the infinite cyclic group generated by $\mathfrak{p}$ and the compact group O^* consisting of all elements of norm 1. Therefore the group Π of all (not necessarily unitary) characters of $\mathbf{K}^*$ is the direct product of the multiplicative group C^* of complex numbers $\lambda \neq 0$ and the group $\hat{O}^*$ of all characters θ of O^*. Thus, every character π on $\mathbf{K}^*$ can be given by a pair (λ, θ), where a $\lambda \in C^*$, $\theta \in \hat{O}^*$.

Every element $x \in \mathbf{K}^*$ has a unique expression in the form

$$x = \mathfrak{p}^k y,$$

where $y \in O^*$. The value of the character π on x is equal to

$$\pi(x) = \lambda^k \theta(y), \tag{1}$$

The following expression for π, which is equivalent to (1), is also convenient. We extend the character θ to the whole of $\mathbf{K}^*$ by setting $\theta(\mathfrak{p}) = 1$. Then we have

$$\pi(x) = |x|^s \, \theta(x), \tag{1'}$$

where s is a complex number connected with λ by the relation $\lambda = |\mathfrak{p}|^s = q^{-s}$.

We note that the set $\hat{O}^*$ of characters θ is countable and discrete so that Π is the union of a countable number of complex planes with zero deleted.

From the integral representation of the Gamma-function

$$\Gamma(\pi) = \int \chi(x) \pi(x) \, d^*x \tag{2}$$

we can derive the expansion of the Gamma-function in a Laurent series in λ. For this purpose we represent $\mathbf{K}^*$ as the union of the sets $\mathfrak{p}^k O^*$, $k = 0, \pm 1, \pm 2, \ldots$.

Then we find

$$\Gamma(\pi) = \sum_k \int_{|y|=1} \chi(\mathfrak{p}^k y) \pi(\mathfrak{p}^k y) \, dy$$

$$= \sum_k \lambda^k \int_{|y|=1} \chi(\mathfrak{p}^k y) \theta(y) \, dy.$$

Thus the coefficients of the expansion of $\Gamma(\pi)$ in a Laurent series in λ are of the form

$$\Gamma_k(\theta) = \int_{|y|=1} \chi(\mathfrak{p}^k y) \theta(y) \, dy. \tag{3}$$

Observe that the integrals (3) converge, in contrast to the integral (2), which must be understood in the sense of generalized functions.

We shall see presently that almost all coefficients $\Gamma_k(\theta)$ can be computed explicitly. From this computation it follows that $\Gamma(\pi) \equiv \Gamma(\lambda, \theta)$ is a rational function of λ for every fixed θ.

First we recall that we subjected the character χ that occurs in the definition of the Gamma-function to the following condition: $\chi(x) = 1$, when $|x| \leq 1$; $\chi(x) \not\equiv 1$ on the set $|x| \leq q$.

We now introduce the concept of the rank of a character θ. We consider the group O^* of elements of norm 1. This group has a decreasing sequence of open subgroups O_n^* consisting of the elements of the form $1 + \mathfrak{p}^n x$, $|x| \leq 1$. From the continuity of θ it follows that when n is sufficiently large, $\theta(x) \equiv 1$ on O_n^*. We define the *rank* of θ as the smallest number n for which $\theta(x) \equiv 1$ on O_n^*. Obviously, the set of characters of a rank not exceeding n is finite.† In particular, there is only one character of rank 0, namely $\theta_0 \equiv 1$.

We now prove the following proposition.

If the rank of the character θ is equal to m, $m > 0$, then $\Gamma_k(\theta) = 0$ for $k \neq -m$; moreover,

$$|\Gamma_{-m}(\theta)| = q^{-m/2}. \tag{4}$$

If the rank of character θ equal to zero, that is, $\theta(x) \equiv 1$, then

$$\Gamma_k(\theta) = \begin{cases} 0 & \text{for } k < -1 \\ -q^{-1} & \text{for } k = -1 \\ 1 - q^{-1} & \text{for } k > -1. \end{cases} \tag{5}$$

Thus, on a disconnected field $\mathbf{K}$ the Gamma-function is of a very simple type. Specifically, if $\theta(x) \not\equiv 1$, then

$$\Gamma(\lambda, \theta) = \Gamma_{-m}(\theta)\lambda^{-m}, \tag{6}$$

where m is the rank of θ $(m > 0)$, and $|\Gamma_{-m}(\theta)| = q^{-m/2}$. If $\theta = \theta_0 \equiv 1$, then

$$\Gamma(\lambda, \theta_0) = (1 - q^{-1}) \sum_{k=0}^{\infty} \lambda^k - q^{-1}\lambda^{-1} = \frac{1 - q^{-1}\lambda^{-1}}{1 - \lambda}. \tag{7}$$

Proof. We begin by discussing the case when θ is a character of rank $m > 0$. Let us show that then $\Gamma_k(\theta) = 0$ for $k \neq -m$.

If $k \geq 0$, then we have, since $\chi(\mathfrak{p}^k y) \equiv 1$,

$$\Gamma_k(\theta) = \int_{|y|=1} \theta(y)\, dy = 0.$$

Now take $k < 0$. We write y in the form

$$y = \alpha_0 + \alpha_1\mathfrak{p} + \ldots + \alpha_n\mathfrak{p}^n + \ldots$$

(see § **1**). Since the rank of θ is m, $\theta(y)$ depends only on $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$, and the dependence on α_{m-1} is nontrivial. On the other hand, the function $\chi(\mathfrak{p}^k y)$ depends only on $\alpha_0, \alpha_1, \ldots, \alpha_{-k-1}$, and the dependence on α_{-k-1} is nontrivial.

If $0 > k > -m$, then $-k - 1 < m - 1$, and so $\chi(\mathfrak{p}^k y)$ does not depend on α_{m-1}. When we split the integrals (3) into integrals over

† This set is the group dual to the finite group O^*/O_n^*.

the cosets of O^*_{m-1}, we find that each of these integrals is equal to zero. Thus, if $0 > k > -m$, then $\Gamma_k(\theta) = 0$.

Finally, if $k < -m$, then $m - 1 < -k - 1$, and so $\theta(y)$ does not depend on α_{-k-1}. When we split the integrals (3) into integrals over sets of the form $y + \mathfrak{p}^{-k-1}O$, and bear in mind that the character $\chi(x)$ is nontrivial on $\mathfrak{p}^{-1}O$, we see again that each of these integrals is equal to zero.† So if $k < -m$, then $\Gamma_k(\theta) = 0$.

We have now shown that if θ is a character of rank $m > 0$, then $\Gamma_k(\theta) = 0$ for $k \neq -m$. It remains to compute $\Gamma_{-m}(\theta)$.

For this purpose we consider the function

$$\varphi(x) = \begin{cases} \theta(x), & \text{when } |x| = 1, \\ 0, & \text{when } |x| \neq 1, \end{cases} \tag{8}$$

and compute its Fourier transform. Let $u = \mathfrak{p}^k v$, $|v| = 1$; then

$$\tilde{\varphi}(u) = \int_K \varphi(x)\chi(ux)\,dx = \int_{|y|=1} \theta(y)\chi(\mathfrak{p}^k v y)\,dy = \Gamma_k(\theta)\theta^{-1}(v).$$

But $\Gamma_k(\theta)$ is different from zero only for $k = -m$. So we have

$$\tilde{\varphi}(u) = \begin{cases} \Gamma_{-m}(\theta)\theta^{-1}(v), & \text{when } u = \mathfrak{p}^{-m}v,\ |v| = 1, \\ 0, & \text{when } |u| \neq q^m. \end{cases} \tag{9}$$

By substituting this value of $\tilde{\varphi}$ in the Plancherel formula for the Fourier transform we obtain the required equation $|\Gamma_{-m}(\theta)|^2 = q^{-m}$.

Now we proceed to the case $\theta = \theta_0 \equiv 1$. The integral to be investigated has the form

$$\begin{aligned} \Gamma_k(\theta_0) &= \int_{|y|=1} \theta_0(y)\,\chi(\mathfrak{p}^k y)\,dy = \int_{|y|=1} \chi(\mathfrak{p}^k y)\,dy \\ &= \int_{|y|\leqslant 1} \chi(\mathfrak{p}^k y)\,dy - \int_{|y|<1} \chi(\mathfrak{p}^k y)\,dy \\ &= q^k \int_{|y|\leqslant q^{-k}} \chi(y)\,dy - q^k \int_{|y|\leqslant q^{-k-1}} \chi(y)\,dy. \end{aligned}$$

We note that

$$\int_{|y|\leqslant q^{-1}} \chi(y)\,dy = \begin{cases} q^{-k} & \text{for } k \geq 0 \\ 0 & \text{for } k < 0. \end{cases}$$

The formula (5) for $\Gamma_k(\theta_0)$ follows immediately from this, and the proposition is proved.

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† Note that the splitting of O^* into domains of the form $y = p^{-k-1}O$, which we have used here, is possible only under the condition that $-k - 1 > 0$, that is, for $k < -1$. This condition is automatically satisfied for $m > 0$, because we have assumed that $k < -m$.

We mention that the relation (4) may be presented in the following form. If $\pi(x) = |x|^s \theta(x)$, where $\theta(\mathfrak{p}) = 1$, and the rank of θ is $m > 0$, then

$$|\Gamma(\pi)| = q^{m(\mathrm{Re}\, s - \frac{1}{2})}.$$

Thus, in this instance it follows that on the set of characters of the form $\pi(x) = |x|^{\frac{1}{2}+rP} \theta(x)$ we have

$$|\Gamma(\pi)| = 1.$$

(The character $\pi_{\frac{1}{2}+rp}(x) = |x|^{\frac{1}{2}+rp}$ is not excluded, because by (7) we have $|\Gamma(\pi_{\frac{1}{2}+rp})| = 1$.)

Now we show that *if the rank of θ is m, $m > 0$, then the following relation holds*:

$$\Gamma_{-m}(\theta)\,\Gamma_{-m}(\theta^{-1}) = q^{-m}\theta(-1). \tag{10}$$

To prove this we consider the function $\varphi(x)$ defined by (8). Its Fourier transform $\tilde{\varphi}(u)$ is expressed by (9). We now compute the Fourier transform $\tilde{\tilde{\varphi}}(x)$ of $\tilde{\varphi}(u)$. Let $x = \mathfrak{p}^k y$, $|y| = 1$. Then

$$\begin{aligned} \tilde{\tilde{\varphi}}(x) &= \int_K \tilde{\varphi}(u)\chi(ux)\, du \\ &= \Gamma_{-m}(\theta) q^m \int_{|v|=1} \theta^{-1}(v)\,\chi(\mathfrak{p}^{k-m} v y)\, dv \\ &= \Gamma_{-m}(\theta) q^m\, \Gamma_{k-m}(\theta^{-1})\,\theta(y) \end{aligned}$$

But $\Gamma_{k-m}(\theta^{-1})$ is different from zero only for $k = 0$. So we have

$$\tilde{\tilde{\varphi}}(x) = \begin{cases} \Gamma_{-m}(\theta)\,\Gamma_{-m}(\theta^{-1}) q^m \theta(x), & \text{when } |x| = 1 \\ 0 & \text{when } |x| \neq 1 \end{cases}$$

that is,

$$\tilde{\tilde{\varphi}}(x) = \Gamma_{-m}(\theta)\,\Gamma_{-m}(\theta^{-1}) q^m \varphi(x) \tag{11}$$

On the other hand, from general properties of the Fourier transform it follows that

$$\tilde{\tilde{\varphi}}(x) = \varphi(-x) = \theta(-1)\,\varphi(x) \tag{12}$$

Comparing (11) and (12) we obtain the required relation (10).

The relation (10) enables us to compute the value

$$\Gamma(\pi) = \Gamma(\lambda, \theta),$$

to within its sign, when $\theta^2 \equiv 1$, and $\theta \not\equiv 1$. For in this case the rank of θ is 1, and by (10), we have

$$\Gamma_{-1}^2(\theta) = q^{-1}\theta(-1),$$

whence $\Gamma_{-1}(\theta) = \pm\sqrt{\theta(-1)}\,q^{-\frac{1}{2}}$. Consequently, by (6),

$$\Gamma(\lambda, \theta) = \pm\sqrt{\theta(-1)}\,q^{-\frac{1}{2}}\lambda^{-1}. \tag{13}$$

The values of $\Gamma(\pi)$ for certain special values of the character π are as follows:

1. $\pi(x) = |x|^s$. In this case $\lambda = q^{-s}$, $\theta \equiv 1$. Consequently, by (7)

$$\Gamma(\pi) = \frac{1 + q^{s-1}}{1 - q^{-s}}. \tag{14}$$

2. $\pi(x) = |x|^s \operatorname{sign}_\varepsilon x$. In this case† $\lambda = -q^{-s}$, $\theta \equiv 1$. Consequently,

$$\Gamma(\pi) = \frac{1 + q^{s-1}}{1 + q^{-s}}. \tag{15}$$

3. $\pi(x) = |x|^s \sin_\tau x$, where $\tau = \mathfrak{p}$ or $\varepsilon\mathfrak{p}$. In this case $\lambda = q^{-s}$, $\theta(y) = \operatorname{sign}_\tau y$, that is, $\theta^2 \equiv 1$. Consequently, by (13) we have

$$\Gamma(\pi) = \pm\sqrt{\operatorname{sign}_\tau(-1)}\, q^{s-\frac{1}{2}}. \tag{16}$$

We give two other relations for the Gamma-function. These can be obtained as consequences of (10). Let π be a character of rank $m > 0$ on O^*. Then we have

$$\Gamma(\pi)\Gamma(\pi^{-1}) = q^{-m}\pi(-1). \tag{17}$$

Further, by comparing this relation with

$$\Gamma(\pi\pi_0)\Gamma(\pi^{-1}) = \pi(-1),$$

where $\pi_0(x) = |x|$ (see **§ 25** (4)), we obtain

$$\Gamma(\pi\pi_0) = q^m\Gamma(\pi). \tag{18}$$

The relation (18) may be regarded as an analogue of the relation $\Gamma(x-1) = x\Gamma(x)$ for the classical Gamma-function.

In the excluded case, when $\pi(x) \equiv 1$ on O^*, that is,

$$\pi(x) = |x|^s,$$

we obtain from (14):

$$\Gamma(\pi)\Gamma(\pi^{-1}) = \frac{(1 - q^{s-1})(1 - q^{-s-1})}{(1 - q^{-s})(1 - q^s)}; \tag{17'}$$

$$\Gamma(\pi\pi_0) = \frac{(1 - q^{-s})(1 - q^s)}{(1 - q^{s-1})(1 - q^{-s-1})}\Gamma(\pi). \tag{18'}$$

We now introduce the concept of the *incomplete Gamma-function*, and define it by the formula:

$$\Gamma^{(k)}(\pi) \equiv \Gamma^{(k)}(\lambda, \theta) = \int\limits_{|x| \leqslant q^k} \chi(x)\pi(x)\, d^*x.$$

On the basis of (4) and (5) we have the following result. *If the rank of the character θ is equal to $m > 0$, then*

$$\Gamma^{(k)}(\pi) = \begin{cases} 0 & \text{for } k < m, \\ \Gamma(\pi) & \text{for } k \leq m. \end{cases}$$

---

† The case 2 is obtained from 1 when s is replaced by $s + \dfrac{2\pi i}{\ln q}$.

In the excluded case, when $\theta = \theta_0 \equiv 1$,

$$\Gamma^{(k)}(\pi) = \begin{cases} \dfrac{(1 - q^{-1})\lambda^k}{1 - \lambda} & \text{for } k \le 0 \\[2ex] \dfrac{1 - q^{-1}\lambda^{-1}}{1 - \lambda} & \text{for } k > 0. \end{cases}$$

So we see that every fixed π the sequence

$$\Gamma^{(0)}(\pi),\ \Gamma^{(1)}(\pi), \dots,\ \Gamma^{(k)}(\pi), \dots$$

stabilizes at a sufficiently large index k.

7. The Integral $\int \chi(ut\bar{t})\, dt$. In what follows we need the integral

$$F(u) = \int \chi(ut\,\bar{t})\; dt\,, \tag{1}$$

where the integration is taken over the plane $\mathbf{K}(\sqrt{\tau})$, which we shall now compute. First of all, (1) can be rewritten as an integral over $\mathbf{K}$ (see the formula on p. 136):

$$\begin{aligned} F(u) &= a_\tau \int\limits_{\operatorname{sign}_\tau x = 1} \chi(ux)\; dx \\ &= \frac{a_\tau}{2} \int\limits_K \chi(ux)\; dx + \frac{a_\tau}{2} \int\limits_K \chi(ux) \operatorname{sign}_\tau x\; dx, \end{aligned}$$

where $a_\tau = 2(1 + q^{-1})(1 + |\tau|)^{-1}$. According to **§ 2.5** we have

$$\int \chi(ux)\; dx = \delta(u), \qquad \int \chi(ux) \operatorname{sign}_\tau x\; dx = \Gamma(\pi)\, \frac{\operatorname{sign}_\tau u}{|u|}\,,$$

where $\pi(x) = |x| \operatorname{sign}_\tau x$. Thus,

$$\int \chi(ut\bar{t})\; dt = c_\tau^{-1}\, \frac{\operatorname{sign}_\tau u}{|u|} + \frac{a_\tau}{2}\, \delta(u), \tag{2}$$

where we have set

$$c_\tau^{-1} = \frac{a_\tau}{2}\, \Gamma(\pi) = \frac{1 + q^{-1}}{1 + |\tau|} \int \chi(x) \operatorname{sign}_\tau x\; dx. \tag{3}$$

Note that the coefficient c_τ satisfies the relation

$$c_\tau = c_\tau \operatorname{sign}_\tau(-1). \tag{4}$$

Hence c_τ is real when $\operatorname{sign}_\tau(-1) = 1$, c_τ is purely imaginary when $\operatorname{sign}_\tau(-1) = -1$.

The coefficient c_τ can be calculated to within its sign on the basis of the results in **§ 2.6**. For by **§ 2.6** (15) we have

$$c_\tau = 1 \quad \text{when} \quad \tau = \varepsilon \tag{5}$$

and by **§ 2.6** (16) we have

$$c_\tau = \pm[\operatorname{sign}_\tau(-1)]^{1/2} q^{-1/2} \quad \text{when} \quad \tau = \mathfrak{p} \text{ or } \varepsilon\mathfrak{p} \tag{6}$$

8. On Functions Resembling Analytic Functions in the Upper and the Lower Half-Plane. Let $\mathbf{K}(\sqrt{\tau})$ be a quadratic extension of a disconnected field $\mathbf{K}$. We define the *upper half-plane* of $\mathbf{K}(\sqrt{\tau})$ as the set of points $z = x + \sqrt{\tau}y$, $\operatorname{sign}_\tau y = 1$; and the *lower half-plane* as the set of points $z = x + \sqrt{\tau}y$, $\operatorname{sign}_\tau y = -1$.

It is easy to verify that *the upper and the lower half-planes are homogeneous spaces relative to the group of fractional-linear transformations* $z' = \dfrac{\alpha z + \gamma}{\beta z + \delta}$, $\alpha\delta - \beta\gamma = 1$.

For disconnected fields the concept of a complex-valued function analytic in the upper or the lower half-plane does not exist. However, we may introduce the concept of a function resembling an analytic function.

With this aim we introduce on $\mathbf{K}$ generalized functions analogous to $(x + i0)^{-1}$ and $(x - i0)^{-1}$ in the case of the real field.

We define the generalized function $(x + \sqrt{\tau}\,0)^{-1}$ as the Fourier transform of the generalized function

$$f_\tau^+(u) = \tfrac{1}{2}(1 + \operatorname{sign}_\tau u),$$

which is equal to 1 when $\operatorname{sign}_\tau u = 1$, and to 0 when $\operatorname{sign}_\tau u = -1$. Similarly we define the generalized function $(x - \sqrt{\tau}\,0)^{-1}$ as the Fourier transform of the generalized function

$$f_\tau^-(u) = \tfrac{1}{2}(1 - \operatorname{sign}_\tau u),$$

which is equal to 1 when $\operatorname{sign}_\tau u = -1$, and to 0 when $\operatorname{sign}_\tau u = 1$.

On the basis of results in **§ 2.6** and **§ 2.7** we may express these functions in terms of the generalized functions $\delta(x)$ and $\dfrac{\operatorname{sign}_\tau x}{|x|}$:

$$(x + \sqrt{\tau}\,0)^{-1} = \tfrac{1}{2}\delta(x) + a_\tau^{-1}c_\tau^{-1}\frac{\operatorname{sign}_\tau x}{|x|}, \tag{1}$$

$$(x - \sqrt{\tau}\,0) = \tfrac{1}{2}\delta(x) - a_\tau^{-1}c_\tau^{-1}\frac{\operatorname{sign}_\tau x}{|x|}, \tag{2}$$

The coefficients c_τ were calculated in 7.

We call a function $f(x)$ *resembling an analytic function in the upper half-plane* if its convolution with $(x - \sqrt{\tau}\,0)^{-1}$ is identically zero:

$$\int (t - \sqrt{\tau}\,0)^{-1} f(x - t)\, dt = 0$$

(or, what is equivalent, if its Fourier transform is concentrated on the half-line $\text{sign}_\tau u = 1$).

The concept of a function resembling an analytic function in the lower half-plane is defined similarly.

9. The Mellin Transform. We define the Mellin transform of a function $f(x)$ by the formula

$$F(\pi) = \int \pi(x) f(x) \, d^*x, \tag{1}$$

where π ranges over the unitary multiplicative characters,† $d^*x = |x|^{-1} \, dx$.

Thus, the Mellin transform may be regarded as the Fourier transform on the multiplicative group $\mathbf{K}^*$ of $\mathbf{K}$.

The Mellin transform is defined for every function $f(x)$ for which

$$\int |f(x)|^2 \, d^*x < \infty.$$

The integral (1) must be understood in the sense of the mean square value.

The inversion formula

$$f(x) = c \int \pi^{-1}(x) F(\pi) \, d\pi \tag{2}$$

and the Plancherel formula

$$\int |f(x)|^2 \, d^*x = c \int |F(\pi)|^2 \, d\pi \tag{3}$$

are valid. The integration is taken here with respect to the invariant measure $d\pi$ on the character group; c is a positive constant depending on the normalization of $d\pi$, which we shall take in what follows so that $c = 1$.

In studying the Mellin transform we have to take as the space of test functions not S, but another space S^*, which we now define.

By S^* we denote the set of functions $f(x)$ satisfying the following requirements:

1. The function $f(x)$ is finite on $\mathbf{K}^*$; in other words, there are positive numbers a and b, $a > b$, such that $f(x) = 0$ for $|x| > a$ and for $|x| < b$.

2. There exists a sufficiently small open subgroup of $\mathbf{K}^*$ such that $f(x)$ is constant on its cosets. In other words,

$$f(xa) = f(x), \tag{4}$$

if the norm $|1 - a|$ is sufficiently small.

† That is, $|\pi(x)| = 1$.

A topology in S^* is introduced in the natural manner.

It is not hard to verify that S^* consists of precisely those functions $f(x)$ for which

$$f(x) \in S \quad \text{and} \quad f(x^{-1}) \in S.$$

Now we define the Mellin transform of a generalized function. As the basis of this definition we take the Plancherel formula (3). Using the notation

$$(\varphi(x), f(x)) = \int \varphi(x) f(x)\, d^*x.$$

$$(\Phi(\pi), F(\pi)) = \int \Phi(\pi) F(\pi)\, d\pi,$$

we can rewrite the Plancherel formula in the form

$$(\varphi(x), \overline{f(x)}) = (\Phi(\pi), \overline{F(\pi)}). \tag{5}$$

Observe that the Mellin transform of $\overline{f(x^{-1})}$ is $\overline{F(\pi)}$. Consequently, by replacing in (5) $f(x)$ by $\overline{f(x^{-1})}$ we find

$$(\varphi(x), f(x^{-1})) = (\Phi(\pi), F(\pi)). \tag{6}$$

Formula (6) defines the Mellin transform as a functional in the function space of the test functions. And so we take this formula as definition of the Mellin transform of a generalized function.

Definition of the Generalized Function $\Gamma(\pi)$. We use the name *generalized Gamma-function* for the Mellin transform of the generalized function $\chi(x)$. Thus, formally, $\Gamma(\pi)$ can be written as an integral

$$\Gamma(\pi) = \int \pi(x) \chi(x)\, |x|^{-1}\, dx. \tag{7}$$

Definition of the Generalized Bessel Function $J(\pi; u)$. We use the name *generalized Bessel function* $J(\pi; u)$ for the Mellin transform of the generalized function $\chi(u(x + x^{-1}))$, that is,

$$J(\pi; u) = \int \pi(x) \chi(u(x + x^{-1}))\, |x|^{-1}\, dx. \tag{8}$$

We write down another integral representation of $J(\pi; u)$. We know that $\chi(t)$ is inverse Mellin transform of $\Gamma(\pi)$, that is,

$$\chi(t) = \int \Gamma(\pi_1) \pi_1^{-1}(t)\, d\pi_1.$$

It follows that

$$\chi(ux) = \int \Gamma(\pi_1) \pi_1^{-1}(u) \pi_1^{-1}(x)\, d\pi_1,$$

$$\chi(ux^{-1}) = \int \Gamma(\pi_2) \pi_2^{-1}(u) \pi_2(x)\, d\pi_2,$$

and therefore,

$$\chi(u(x + x^{-1})) = \int \Gamma(\pi_1)\,\Gamma(\pi_2)\,\pi_1^{-1}\pi_2^{-1}(u)\,\pi_1^{-1}\pi_2(x)\;d\pi_1\,d\pi_2.$$

Substituting this expression in (8) for $J(\pi; u)$ and integrating with respect to x and to π_1 we find

$$J(\pi; u) = \int \Gamma(\pi'^{-1})\,\Gamma(\pi\pi'^{-1})\,\pi^{-1}\pi'^{2}(u)\;d\pi'. \tag{9}$$

10. The Relation Between the Gamma-Function Connected with the Ground Field K and the Gamma-Function Connected with the Quadratic Extension $\mathbf{K}(\sqrt{\tau})$ of K. Apart from the Gamma-function connected with **K** we consider the Gamma-function $\Gamma_\tau(\pi)$, connected with the field $\mathbf{K}(\sqrt{\tau})$:

$$\Gamma_\tau(\pi) = \int\limits_{\mathbf{K}(\sqrt{\tau})} \chi_\tau(t)\,\pi(t)\;d^*t, \tag{1}$$

where π ranges over the set of multiplicative characters on $\mathbf{K}(\sqrt{\tau})$. We assume that the additive character $\chi_\tau(t)$ on $\mathbf{K}(\sqrt{\tau})$ is given by the following formula:

$$\chi_\tau(t) = \chi(t + \bar{t}), \tag{2}$$

where χ is the given additive character (of rank 0) on **K**. We note that the rank of $\chi_\tau(t)$ on $\mathbf{K}(\sqrt{\tau})$ is zero when $\tau = \varepsilon$, and 1 when $\tau = \mathfrak{p}$ or $\varepsilon\mathfrak{p}$.

We show that *the following relation holds:*

$$\Gamma_\tau(\pi\bar{\pi}) = |\tau|^{-1}\,c_\tau\,\Gamma(\pi)\,\Gamma(\pi\pi_\tau), \tag{3}$$

where $\bar{\pi}(t)$ denotes the character $\bar{\pi}(t) = \pi(\bar{t})$

$$\pi_\tau(x) \equiv \operatorname{sign}_\tau x \quad\text{and}\quad c_\tau^{-1} = \frac{a_\tau}{2}\int \chi(x)\,\pi_\tau(x)\;dx.$$

We set

$$f(\pi) = \frac{\Gamma(\pi)\,\Gamma(\pi\pi_\tau)}{\Gamma_\tau(\pi\bar{\pi})}. \tag{4}$$

Since $\Gamma_\tau^{-1}(\pi\bar{\pi}) = |\tau|\;\Gamma_\tau(\pi^{-1}\bar{\pi}^{-1}\pi_0^2)$, where $\pi_0(t) = |t\bar{t}|^{1/2}$ (see § 2.5 (4)), we have

$$\begin{aligned} f(\pi) &= |\tau|\;\Gamma(\pi)\,\Gamma(\pi\pi_\tau)\,\Gamma_\tau(\pi^{-1}\bar{\pi}^{-1}\pi_0^2) \\ &= |\tau|\int \chi(x + y + t + \bar{t})\,\pi\!\left(\frac{xy}{t\bar{t}}\right)\operatorname{sign}_\tau y\;|x|^{-1}\,|y|^{-1}\;dx\,dy\,dt; \end{aligned}$$

the integration is taken with respect to the variables $x, y \in \mathbf{K}$ and $t \in \mathbf{K}(\sqrt{\tau})$. By making the change of variables $x = \dfrac{t\bar{t}}{y}s$ and $t = yt'$

the integral reduces to the form

$$f(\pi) = |\tau|\int \chi(y(st\bar{t} + 1 + t + \bar{t}))\pi_0\pi_\tau(y)\pi\pi_0^{-1}(s)\,dy\,ds\,dt$$
$$= \int \chi(y(st\bar{t} + 1 - s^{-1}))\pi_0\pi_\tau(y)\pi\pi_0^{-1}(s)\,dy\,ds\,dt.$$

(The transition to the last integral is effected by the substitution: $t = t' - s^{-1}$.) Integrating with respect to y, we find

$$f(\pi) = |\tau|\;\Gamma(\pi_0^2\pi_\tau)\int \pi_0^{-2}\pi_\tau(st\bar{t} + 1 - s^{-1})\pi\pi_0^{-1}(s)\,dt\,ds$$
$$= |\tau|\;\Gamma(\pi_0^2\pi_\tau)\int \pi_0^{-2}\pi_\tau(t\bar{t} + s - 1)\pi\pi_0^{-1}\pi_\tau(s)\,dt\,ds. \tag{5}$$

(Change of variable $t = s^{-1}t'$.)

We compute separately the integral

$$\varphi(x) = \int \pi_0^{-2}\pi_\tau(t\bar{t} + x)\,dt.$$

Passing from $\varphi(x)$ to its Fourier transform we obtain

$$\tilde{\varphi}(u) = \int \chi(ux)\pi_0^{-2}\pi_\tau(t\bar{t} + x)\,dt\,dx$$
$$= \int \chi(ux)\pi_0^{-2}\pi_\tau(x)\,dx\int \chi(-ut\bar{t})\,dt$$
$$= \Gamma(\pi_0^{-1}\pi_\tau)\pi_0\pi_\tau(u)\left[c_\tau^{-1}\pi_0^{-1}\pi_\tau(-u) + \frac{a_\tau}{2}\,\delta(u)\right]$$

(see **§ 2.7** (2)).

So we have

$$\tilde{\varphi}(u) = c_\tau^{-1}\,\Gamma(\pi_0^{-1}\pi_\tau)\pi_\tau(-1),$$

hence,

$$\varphi(x) = c_\tau^{-1}\,\Gamma(\pi_0^{-1}\pi_\tau)\pi_\tau(-1)\delta(x).$$

Thus, we have established that

$$\int \pi_0^{-2}\pi_\tau(t\bar{t} + s - 1)\,dt = c_\tau^{-1}\,\Gamma(\pi_0^{-1}\pi_\tau)\pi_\tau(-1)\;\delta(s - 1).$$

Substituting this expression in (5) we find that

$$f(\pi) = |\tau|\,c_\tau^{-1}\,\Gamma(\pi_0^{-1}\pi_\tau)\Gamma(\pi_0^2\pi_\tau)\pi_\tau(-1).$$

Finally, observing that

$$\Gamma(\pi_0^{-1}\pi_\tau) = \Gamma(\pi_0\pi_\tau), \quad \Gamma(\pi_0^2\pi_\tau) = \Gamma^{-1}(\pi_0\pi_\tau)\pi_\tau(-1),$$

we obtain the end result: $f(\pi) = |\tau|\,c_\tau^{-1}$, that is

$$\Gamma_\tau(\pi\bar{\pi}) = |\tau|^{-1}\,c_\tau\,\Gamma(\pi)\,\Gamma(\pi\pi_\tau).$$

§ 3. IRREDUCIBLE REPRESENTATIONS OF THE GROUP OF MATRICES OF ORDER 2 WITH ELEMENTS FROM A LOCALLY COMPACT FIELD (THE CONTINUOUS SERIES)

In this and the subsequent sections we study representations of the group G of matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha\delta - \beta\gamma = 1$, whose elements α, β, γ, δ belong to a locally compact topological field. In **§ 3** we give a description of the continuous series of irreducible unitary representations of G. Other (discrete) series of irreducible unitary representations of G will be discussed in **§ 4**. In **§ 5** we compute the traces (characters) of the irreducible representations of G, and in **§ 6** we obtain an expansion of a function $f(g)$ on G as a Fourier integral (Plancherel's theorem).

Instead of G we frequently consider groups related to it:

1. The factor group $G_1 = G/\mathfrak{Z}$ of G over its center $\mathfrak{Z}$ ($\mathfrak{Z}$ consists of the two elements e and $-e$, where e is the unit matrix).

2. The group G_2 of all fractional-linear transformations $x' = \dfrac{\alpha x + \gamma}{\beta x + \delta}$.

It is easy to show that G_2 is isomorphic to the group of all automorphisms of G, and that G_1 is isomorphic to the group of all inner automorphisms of G. Thus, G_1 is a (normal) subgroup of G_2. It is not hard to show that

$$G_2/G_1 \cong \mathbf{K}^*/(\mathbf{K}^*)^2.$$

Thus, $G_2 = G_1$ when $\mathbf{K}$ is the field of complex numbers, $G_2 : G_1 = 2$ when $\mathbf{K}$ is the field of real numbers, $G_2 : G_1 = 4$ when $\mathbf{K}$ is a disconnected field.†

All the results to be expounded below carry over to the groups G_1 and G_2 without any essential modifications.

1. The Continuous Series of Unitary Representations of G. We begin with a description of the continuous series of representations of G. For the field of complex numbers this series of representations was discovered by Gel'fand and Naimark. The construction given by them carries over directly to the case of a locally compact topological field $\mathbf{K}$.

A representation of the continuous series is given by a unitary multiplicative character $\pi(x)$ *on* $\mathbf{K}$.

The representation is constructed in the space of complex-valued functions $\varphi(x)$ on $\mathbf{K}$ for which

$$(\varphi, \varphi) = \int |\varphi(x)|^2 \, dx < \infty.$$

† The special case when the characteristic of the finite field O/P associated with $\mathbf{K}$ is 2 is excluded (see **§ 1.5**).

The representation operator T_π corresponding to the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ has the following form:

$$T_\pi(g)\,\varphi(x) = \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)\pi(\beta x + \delta)\,|\beta x + \delta|^{-1}. \qquad (1)$$

The fact that the operators $T_\pi(g)$ form a representation, that is, that $T_\pi(g_1 g_2) = T_\pi(g_1)\,T_\pi(g_2)$, is established by direct verification.

A similar construction of representations is available for the group of matrices with elements from a *finite* field $\mathbf{K}_q$ of order q. To describe these representations it is convenient to go over from the functions $\varphi(x)$ to homogeneous functions $f(x_1, x_2)$ of two variables. Then we have the following description of the representations. Each representation is given by a multiplicative character $\pi(t)$ on $\mathbf{K}_q$. It is constructed in the space of functions $f(x_1, x_2)$ on $\mathbf{K}_q$, $(x_1, x_2) \neq (0, 0)$, that satisfy the condition of homogeneity

$$f(tx_1, tx_2) = \pi(t) f(x_1, x_2).$$

The representation operator $T_\pi(g)$ has the form

$$T_\pi(g) f(x_1, x_2) = f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2). \qquad (2)$$

These representations are irreducible except when $\pi \equiv 1$ (in this case a one-dimensional representation splits off), and when $\pi(t)$ assumes only the values ± 1 (in this case the representation decomposes into two representations of equal dimension). As in the case of an arbitrary locally compact topological field $\mathbf{K}$, the representations $T_\pi(g)$ and $T_{\pi^{-1}}(g)$ turn out to be equivalent. The traces of the representations $T_\pi(g)$ were first computed by Frobenius.

Apart from the representations (2), the group of matrices over a finite field $\mathbf{K}_q$ also has an "analytic" series of representations, which we shall discuss in § 4.

Let us show that the operators $T_\pi(g)$ are unitary, that is,

$$(T_\pi(g)\,\varphi, T_\pi(g)\,\varphi) = (\varphi, \varphi). \qquad (3)$$

Indeed, we have

$$(T_\pi(g)\,\varphi, T_\pi(g)\,\varphi) = \int \left|\varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)\right|^2 |\beta x + \delta|^{-2}\,dx.$$

By making the change of variable $x' = \dfrac{\alpha x + \gamma}{\beta x + \delta}$ and using the equation $dx' = |\beta x + \delta|^{-2}\,dx$ we obtain equation (3) at once.

Now we give a derivation of the formula $dx' = |\beta x + \delta|^{-2}\,dx$. We set $dx' = a(x, g)\,dx$. An immediate consequence of the definition of $a(x, g)$ is the functional equation

$$a(x, g_1 g_2) = a(x, g_1) a(xg_1, g_2). \qquad (4)$$

(Here xg is the result of applying to x the fractional linear transformation corresponding to g.) The same relation (4) is easily seen to be satisfied by the function $|\beta x + \delta|^{-2}$.

Now we observe that every matrix g can be represented as a product of matrices of the following types:

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \quad \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (5)$$

Therefore, by virtue of (4), the function $a(x, g)$ is uniquely determined by its values on the matrices g of the form (5). So it is sufficient to verify that

$$a(x, g) = |\beta x + \delta|^{-2} \tag{6}$$

for matrices g of the form (5). But for these matrices (6) is obvious. For if $g = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$, then $x' = x + \gamma$, hence, $dx' = dx$. If $g = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$, then $x' = \delta^{-2}x$, hence, $dx' = |\delta|^{-2}\, dx$. Finally, if $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $x' = -\dfrac{1}{x}$; since the multiplicatively invariant measure $d^*x = |x|^{-1}\, dx$ is preserved under this transformation, we have $|x'|^{-1}\, dx' = |x|^{-1}\, dx$, hence $dx' = |x|^{-2}\, dx$.

2. Another Realization of the Representations of the Continuous Series. We obtain another realization of the representations of the continuous series by going over from the functions $\varphi(x)$ to their Fourier transforms

$$\tilde{\varphi}(u) = \int \varphi(x)\chi(-ux)\, dx, \tag{1}$$

Let us find the expression for the operator $T_\pi(g)$ in this realization. The operator $T_\pi(g)$ acts on the function $\tilde{\varphi}(u)$ by the following formula:

$$\begin{aligned} T_\pi(g)\tilde{\varphi}(u) &= \int [T_\pi(g)\varphi(x)]\chi(-ux)\, dx \\ &= \int \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)\pi(\beta x + \delta)\, |\beta x + \delta|^{-1}\, \chi(-ux)\, dx. \end{aligned} \tag{2}$$

It remains for us to express the right-hand side of (2) again in terms of $\tilde{\varphi}(u)$.

By the formula for the inverse Fourier transform we have

$$\varphi(x) = \int \tilde{\varphi}(v)\chi(vx)\, dv.$$

Consequently,

$$T_\pi(g)\tilde{\varphi}(u) = \int \chi\left(-ux + v\,\frac{\alpha x + \gamma}{\beta x + \delta}\right)\pi(\beta x + \delta)\, |\beta x + \delta|^{-1}\, \tilde{\varphi}(v)\, dv\, dx.$$

Henceforth we call this realization of the continuous series the *χ-realization.*

Thus, *in the χ-realization a representation of the continuous series is constructed in the space of functions $\varphi(u)$ for which*

$$(\varphi, \varphi) = \int |\varphi(u)|^2\, du < \infty.$$

The representation operator $T_\pi(g)$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, is given by the formula

$$T_\pi(g)\varphi(u) = \int K_\pi(g \mid u, v)\varphi(v)\, dv, \tag{3}$$

where

$$K_\pi(g \mid u, v) = \int \chi\left(-ux + v\frac{\alpha x + \gamma}{\beta x + \delta}\right)\pi(\beta x + \delta)\,|\beta x + \delta|^{-1}\,dx. \quad (4)$$

Let us examine the expression (4) in detail. To begin with we assume that $\beta \neq 0$. Then it is convenient to write (4) in a somewhat different form, by making the change of variables $\beta x + \delta = t$. After an elementary transformation we find

$$K_\pi(g \mid u, v) = |\beta|^{-1}\chi\left(\frac{\delta u + \alpha v}{\beta}\right)\int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\,|t|^{-1}\,dt. \quad (5)$$

We mention a peculiarity of this formula. It tells us that $K_\pi(g \mid u, v)$ is a product of two functions—the function

$$|\beta|^{-1}\chi\left(\frac{\delta u + \alpha v}{\beta}\right),$$

one and the same for all the representations of the series, and the Bessel function

$$\int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\,|t|^{-1}\,dt,$$

which does not depend essentially on g.

Now we discuss the special case when $\beta = 0$. Then we have

$$\begin{aligned} K_\pi(g \mid u, v) &= \chi\left(\frac{\gamma}{\delta}v\right)\pi(\delta)\,|\delta|^{-1}\int \chi\left(\left(-u + \frac{\alpha}{\delta}v\right)x\right)dx \\ &= \chi\left(\frac{\gamma}{\delta}v\right)\pi(\delta)\,|\delta|^{-1}\,\delta\left(-u + \frac{\alpha}{\delta}v\right). \end{aligned}$$

Bearing in mind that in this case $\alpha = \delta^{-1}$, we may rewrite this formula as follows:

$$K_\pi(g \mid u, v) = \pi(\delta)\,|\delta|\,\chi(\delta\gamma u)\,\delta(v - \delta^2 u). \quad (6)$$

So the operator $T_\pi(g)$ corresponding to the matrix $g = \begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}$ has the following form in the χ-realization:

$$T_\pi(g)\varphi(u) = \pi(\delta)\,|\delta|\,\chi(\delta\gamma u)\,\varphi(\delta^2 u). \quad (7)$$

There is one further convenient realization of the representations of the continuous series, which we call the *π-realization.* We obtain it by going over from the functions $\varphi(x)$ to their Mellin transforms

$$F(\pi_1) = \int \varphi(x)\pi_1(x)\,|x|^{-1/2}\,dx; \quad (8)$$

where π_1 ranges over the unitary multiplicative characters. From

the inversion formula for the Mellin transform (see **§ 2.9**) it follows that

$$\int |\varphi(x)|^2\, dx = \int |F(\pi_1)|^2\, d\pi_1.$$

Let us find the expression for the operator $T_\pi(g)$ in the π-realization. By definition,

$$\begin{aligned} T_\pi(g)F(\pi_1) &= \int [T_\pi(g)\,\varphi(x)]\pi_1(x)\ |x|^{-1/2}\, dx \\ &= \int \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)\pi(\beta x + \delta)\ |\beta x + \delta|^{-1}\ \pi_1(x)\ |x|^{-1/2}\, dx. \end{aligned} \tag{9}$$

It remains to express the right-hand side of (9) again in terms of the function $F(\pi_1)$.

By the inversion formula for the Mellin transform we have

$$\varphi(x) = \int F(\pi_2)\pi_2^{-1}(x)\ |x|^{-1/2}\, d\pi_2.$$

Consequently,

$$\begin{aligned} T_\pi(g)F(\pi_1) = \int \pi_2^{-1}\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)\left|\frac{\alpha x + \gamma}{\beta x + \delta}\right|^{-1/2} \pi(\beta x + \delta) \\ |\beta x + \delta|^{-1}\ \pi_1(x)\ |x|^{-1/2}\, F(\pi_2)\, dx\, d\pi_2. \end{aligned}$$

Thus, *in the π-realization a representation of the continuous series is constructed in the space of functions $F(\pi_1)$ on the group of multiplicative characters π_1 for which*

$$(F, F) = \int |F(\pi_1)|^2\, d\pi_1 < \infty.$$

The representation operator $T_\pi(g)$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, is given by the formula

$$T_\pi(g)F(\pi_1) = \int K_\pi(g \mid \pi_1, \pi_2)F(\pi_2)\, d\pi_2, \tag{10}$$

where

$$\begin{aligned} K_\pi(g \mid \pi_1, \pi_2) = \int \pi\pi_2(\beta x + \delta)\ |\beta x + \delta|^{-1/2} \\ \pi_2^{-1}(\alpha x + \gamma)\ |\alpha x + \gamma|^{-1/2}\ \pi_1(x)\ |x|^{-1/2}\, dx. \end{aligned} \tag{11}$$

The expression (11), which gives the matrix element of the operator $T_\pi(g)$ in the π-representation, can appropriately be called the *hypergeometric function* of π, π_1, π_2. For the field of real numbers $K_\pi(g \mid \pi_1, \pi_2)$ can be expressed immediately in terms of the hypergeometric function of Gauss.

Note that if one of the elements of g is zero, then the hypergeometric function (11) degenerates into a Beta-function. For

example, if $\alpha = 0$, we have

$$K_\pi(g \mid \pi_1, \pi_2) = \pi_2^{-1}(\gamma)\, |\gamma|^{-1/2} \int \pi\pi_2(\beta x + \delta)\, |\beta x + \delta|^{-1/2}\, \pi_1(x)\, |x|^{-1/2}\, dx.$$

The formulae for the representation operators corresponding to the matrices

$$\delta = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}, \qquad z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \qquad z \neq 0,$$

$$\text{and } \zeta = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}, \qquad \zeta \neq 0$$

have a very simple form.

For on the basis of (10) and (11) we find after elementary transformations that

$$T_\pi(\delta)F(\pi_1) = \pi\pi_1^2(\delta)F(\pi_1), \tag{12}$$

$$T_\pi(z)F(\pi_1) = \int \frac{\Gamma(\pi_1\pi_0)}{\Gamma(\pi_2\pi_0)}\, \Gamma(\pi_1^{-1}\pi_2)\pi_1\pi_2^{-1}(z)F(\pi_2)\, d\pi_2, \tag{13}$$

$$T_\pi(\zeta)F(\pi_1) = \int \frac{\Gamma(\pi\pi_2\pi_0)}{\Gamma(\pi\pi_1\pi_0)}\, \Gamma(\pi_1\pi_2^{-1})\pi_1^{-1}\pi_2(-\zeta)F(\pi_2)\, d\pi_2. \tag{14}$$

It is convenient to go over from $T_\pi(g)$ to the equivalent representation $T'_\pi(g) = A^{-1}T_\pi(g)A$, where A is the operator of multiplication by $\Gamma(\pi_1\pi_0)$. Clearly, the kernels of the operators $T'_\pi(g)$ are obtained from those of $T_\pi(g)$ by multiplying by $\dfrac{\Gamma(\pi_2\pi_0)}{\Gamma(\pi_1\pi_0)}$. So we obtain

$$T'_\pi(\delta)F(\pi_1) = \pi\pi_1^2(\delta)F(\pi_1), \tag{15}$$

$$T'_\pi(z)F(\pi_1) = \int \Gamma(\pi_1^{-1}\pi_2)\pi_1\pi_2^{-1}(z)F(\pi_2)\, d\pi_2, \tag{16}$$

$$T'_\pi(\zeta)F(\pi_1) = \int \frac{\Gamma(\pi_2\pi_0)\,\Gamma(\pi\pi_2\pi_0)}{\Gamma(\pi_1\pi_0)\,\Gamma(\pi\pi_1\pi_0)}\, \Gamma(\pi_1\pi_2^{-1})\pi_1^{-1}\pi_2(-\zeta)F(\pi_2)\, d\pi_2. \tag{17}$$

This realization of the representation has the advantage that the operator $T'_\pi(z)$ in it does not depend on the "index" of the representation. Since the matrices z and ζ generate the whole group G, the representation $T'_\pi(g)$ is completely determined by the formula for the operator $T'_\pi(\zeta)$. In this formula the only factor depending on the index of the representation is

$$a(\pi \mid \pi_1, \pi_2) = \frac{\Gamma(\pi_2\pi_0)\,\Gamma(\pi\pi_2\pi_0)}{\Gamma(\pi_1\pi_0)\,\Gamma(\pi\pi_1\pi_0)},$$

under the integral sign, and this then gives us our representation.

In **§ 4.5** we shall show that similar formulae hold for the representations of the discrete series.

3. Equivalence of Representations of the Continuous Series. We are going to show that *the representations of the continuous series* $T_\pi(g)$ *and* $T_{\pi^{-1}}(g)$ *are equivalent.*

To prove this we consider the operator $T_\pi(g)$ in the χ-realization. The kernel $K_\pi(g \mid u, v)$ of $T_\pi(g)$ is given by the following formula:

$$K_\pi(g \mid u, v) = |\beta|^{-1}\chi\left(\frac{\delta u + \alpha v}{\beta}\right)\int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\ |t|^{-1}\, dt.$$

Making the change of variable $t = vu^{-1}t'^{-1}$ under the integral, we obtain

$$K_\pi(g \mid u, v)$$
$$= \frac{\pi(v)}{\pi(u)}\,|\beta|^{-1}\,\chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi^{-1}(t)\ |t|^{-1}\, dt,$$

that is,

$$K_\pi(g \mid u, v) = \frac{\pi(v)}{\pi(u)}\, K_{\pi^{-1}}(g \mid u, v).$$

So we have shown that

$$T_\pi(g) = A^{-1}T_{\pi^{-1}}(g)A,$$

where A is the operator of multiplying by $\pi(u)$:

$$A\varphi(u) = \pi(u)\,\varphi(u).$$

Hence, the representations $T_\pi(g)$ and $T_{\pi^{-1}}(g)$ are equivalent.

In **§ 5** we shall see that there are no other pairs of equivalent representations in the continuous series.

4. Irreducibility of the Representations of the Continuous Series. We show that the representations $T_\pi(g)$ of the continuous series are irreducible, apart from certain special values of π.

We recall that a unitary representation $T(g)$ is called irreducible if the representation space contains no invariant subspace other than zero. The following is an equivalent definition: *a unitary representation* $T(g)$ *is called irreducible if every bounded operator in the representation space that commutes with all the operators* $T(g)$ *is a multiple of the unit operator.*

The following propositions hold.

1. *For the field of complex numbers all the representations* $T_\pi(g)$ *are irreducible.*

2. *For the field of real numbers all the representations* $T_\pi(g)$ *are irreducible, except* $\pi(x) = \operatorname{sign} x$. *In this special case* $\pi(x) = \operatorname{sign} x$, *the representation* $T_\pi(g)$ *splits into two irreducible representations.*

3. *For a disconnected field all the representations* $T_\pi(g)$ *are irreducible, except when* $\pi(x) = \operatorname{sign}_\tau x$, $\tau = \mathfrak{p}$, $\varepsilon\mathfrak{p}$, *or* ε *(see* **§ 1.7**). *In each of these special cases the representation splits into two irreducible representations.*

We give a proof only for the case of a disconnected field; for connected fields the proof is similar.†

We consider the representation $T_\pi(g)$ in the χ-realization. Our aim is to describe all the bounded operators A that commute with the $T_\pi(g)$;

$$AT_\pi(g) = T_\pi(g)A.$$

First, we see what we can derive from the permutability of A with the operators $T_\pi(g)$, where $g = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$. These operators are of the form

$$T_\pi(g)\,\varphi(u) = \chi(\gamma u)\,\varphi(u). \tag{1}$$

Hence, A commutes with the operators of multiplication by $\chi(\gamma u)$, and therefore, with every operator of multiplication by a bounded function. From this it follows that A itself is an operator of multiplication by a bounded function $a(u)$:

$$A\varphi(u) = a(u)\,\varphi(u).$$

Next, we see what we can derive from the permutability of A with the operators $T_\pi(g)$ where $g = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$. These operators are of the form

$$T_\pi(g)\,\varphi(u) = \pi(\delta)\;|\delta|\;\varphi(\delta^2 u). \tag{2}$$

The condition that A commutes with the operators $T_\pi(g)$ can be written in the form

$$a(\delta^2 u) = a(u)$$

for every $\delta \neq 0$. So the function $a(u)$ is constant on every coset of $\mathbf{K}^*/(\mathbf{K}^*)^2$. We can show, in fact, that in the nonexceptional cases $a(u)$ is constant on the whole of $\mathbf{K}$, and that in the exceptional cases it can assume only two distinct values. This then will prove the theorem.

† Another proof of the irreducibility of the representations for the field of complex and the field of real numbers can be found in Gel'fand, et al. [27].

We now write the condition that A commutes with the operator $T_\pi(s)$, where $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This condition takes the form

$$\mathbf{K}_\pi(s \mid u, v)a(v) = a(u)\mathbf{K}_\pi(s \mid u, v), \tag{3}$$

where

$$\mathbf{K}_\pi(s \mid u, v) = \int \chi(ut + vt^{-1})\pi(t)\, d^*t. \tag{4}$$

Let $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$ be distinct cosets of $\mathbf{K}^*/(\mathbf{K}^*)^2$. If we can show that $\mathbf{K}_\pi(s \mid u, v) \not\equiv 0$ when $u \in \mathbf{K}^{(1)}$, $v \in \mathbf{K}^{(2)}$, then it follows from (3) that $a(u)$ assumes identical values on $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$.

Let $u \in \mathbf{K}^{(1)}$, $v \in \mathbf{K}^{(2)}$. We assume that $|u|$, $|v|$ are sufficiently small so that $\chi(ut) \equiv 1$, $\chi(vt) \equiv 1$ for $|t| \leq 1$. Then the expression for $K_\pi(s \mid u, v)$ can be written in the following form:

$$K_\pi(s \mid u, v)$$
$$= \int_{|t|<1} \chi(vt^{-1})\pi(t)\, d^*t + \int_{|t|>1} \chi(ut)\pi(t)\, d^*t + \int_{|t|=1} \pi(t)\, d^*t. \tag{5}$$

First, we take the case $\pi(t) \equiv 1$. Here we find, after a change of variables,

$$K_\pi(s \mid u, v) = \int_{|t|>|v|} \chi(t)\, d^*t + \int_{|t|>|u|} \chi(t)\, d^*t + (1 - q^{-1}).$$

Since $\chi(t) \equiv 1$ when $|t|$ is sufficiently small, this expression cannot possibly be constant, and hence $K_\pi(s \mid u, v) \not\equiv 0$. So we have shown that for $\pi \equiv 1$ the function $a(u)$ is a constant and therefore the representation $T_\pi(g)$ is irreducible.

Now let $\pi(t) \not\equiv 1$. Then we have

$$\int_{|t|\geq 1} \chi(vt^{-1})\pi(t)\, d^*t + \int_{|t|\leq 1} \chi(ut)\pi(t)\, d^*t - \int_{|t|=1} \pi(t)\, d^*t$$
$$= \int \pi(t)\, d^*t = 0.$$

Adding this to (5) we find

$$K_\pi(s \mid u, v) = \int \chi(vt^{-1})\pi(t)\, d^*t + \int \chi(ut)\pi(t)\, d^*t,$$

where the integrals are taken over the whole of K. By a change of variable we obtain

$$K_\pi(s \mid u, v) = \Gamma(\pi^{-1})\pi(v) + \Gamma(\pi)\pi^{-1}(u).$$

Here the coefficients $\Gamma(\pi)$ and $\Gamma(\pi^{-1})$ are different from zero. We assume that $\pi(u)$ is not constant on $(\mathbf{K}^*)^2$. Then $\pi(v)$ is not constant when v ranges over a coset of $\mathbf{K}^*/(\mathbf{K}^*)^2$, and hence $K_\pi(s \mid u, v) \not\equiv 0$. This shows that $a(u)$ is constant and therefore the representations $T_\pi(g)$ are irreducible when $\pi(t) \not\equiv 1$ for $t \in (\mathbf{K}^*)^2$.

Finally, we discuss the special case $\pi(t) = 1$ for $t \in (\mathbf{K}^*)^2$. Such characters π have the form $\pi_\tau(t) = \operatorname{sign}_\tau t$, where $\tau = \mathfrak{p}$, $\varepsilon\mathfrak{p}$, or ε (the case $\pi \equiv 1$ was considered above).

Here we have, because $\pi_\tau = \pi_\tau^{-1}$,

$$\mathbf{K}_\pi(s \mid u, v) = \Gamma(\pi_\tau)[\operatorname{sign}_\tau u + \operatorname{sign}_\tau v].$$

Hence, if $\operatorname{sign}_\tau u = \operatorname{sign}_\tau v$, then $\mathbf{K}_\pi(s \mid u, v) \neq 0$.

This shows that for $\pi(t) = \operatorname{sign}_\tau t$ the function $a(v)$ is constant on the set of elements v where $\operatorname{sign}_\tau v$ is constant, that is, $a(v)$ assumes not more than two distinct values. Hence, the representation $T_\pi(g)$, $\pi(t) = \operatorname{sign}_\tau t$, if it splits at all, splits into no more than two irreducible representations. This fact will be shown in **§ 3.5**.

5. The Decomposition of the Representations $T_{\pi_\tau}(g)$, $\pi_\tau(t) = \operatorname{sign}_\tau t$, **into Irreducible Representations.** We decompose the space of functions $\varphi(u)$ into the direct sum of two subspaces: the subspace H^+ of functions $\varphi(u)$ that are zero for $\operatorname{sign}_\tau u = -1$ and the subspace H^- of functions $\varphi(u)$ equal to zero for $\operatorname{sign}_\tau u = 1$. We show now that *these subspaces H^+ and H^- are invariant under the operators $T_{\pi_\tau}(g)$, $\pi_\tau(t) = \operatorname{sign}_\tau t$.*

First, we recall that every matrix g can be represented as a product of matrices of the form $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$, $\begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$, and the matrix $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore, it is sufficient to verify that H^+ and H^- are invariant under the operators corresponding to these matrices.

Clearly, the operators $T_{\pi_\tau}(g)$ corresponding to the matrices $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ and $\begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$ preserve the spaces H^+ and H^- (because the first reduces to multiplying $\varphi(u)$ by the function $\chi(\gamma u)$ and the second carries $\varphi(u)$ into $\pi(\delta)\,|\delta|\,\varphi(\varphi^2 u)$ (see **§ 3.4** (1) and (2)). Thus, it remains to show the invariance of H^+ and H^- under the operator $T_{\pi_\tau}(s)$:

$$T_{\pi_\tau}(s)\,\varphi(u) = \int \chi(ut + vt^{-1}) \operatorname{sign}_\tau t\,\varphi(v)\, d^*t\, dv.$$

To prove this we pass from the functions $\varphi(u)$ to their Mellin transforms

$$F(\pi) = \int \varphi(u)\, |u|^{1/2}\, \pi(u)\, d^*u.$$

We write down the action of $T_{\pi_\tau}(s)$ on $F(\pi)$:

$$T_{\pi_\tau}(s)F(\pi) = \int \chi(ut + vt^{-1}) \operatorname{sign}_\tau t\varphi(v) \, |u|^{1/2} \pi(u) \, d^*t \, dv \, d^*u.$$

Integrating first with respect to u and then with respect to t we obtain

$$T_{\pi_\tau}(s)F(\pi) = \Gamma(\pi\pi_1)\,\Gamma(\pi\pi_1\pi_\tau) \int \pi^{-1}\pi_1^{-1}\pi_\tau(v)\,\varphi(v)\,dv,$$

where $\pi_1(v) = |v|^{1/2}$. On the right-hand side we substitute for $\varphi(v)$ its expression in terms of the Mellin transform $F(\pi)$:

$$\varphi(v) = \pi_1^{-1}(v) \int \pi'^{-1}(v)F(\pi')\,d\pi'.$$

After integrating with respect to v and to π' we obtain the following formula for $T_{\pi_\tau}(s)$:

$$T_{\pi_\tau}(s)F(\pi) = \Gamma(\pi\pi_1)\,\Gamma(\pi\pi_1\pi_\tau)F(\pi^{-1}\pi_\tau). \tag{1}$$

Let us show that the operator $T_{\pi_\tau}(s)$ defined by (1) preserves the subspaces H^+ and H^-. For this purpose we write down the condition that $F(\pi)$ belongs to H^+. The condition that $\varphi(u)$ belongs to H^+ can be written as follows:

$$\varphi(u)\pi_\tau(u) = \varphi(u),$$

where $\pi_\tau(u) = \operatorname{sign}_\tau u$. Obviously in the Mellin transform this condition can be written:

$$F(\pi\pi_\tau) = F(\pi). \tag{2}$$

Suppose now that $F(\pi)$ belongs to H^+, that is, that it satisfies (2), and let $F_1(\pi) = T_{\pi_\tau}(s)F(\pi)$. Bearing in mind that $\pi_\tau^2 = 1$ we then obtain from (1):

$$\begin{aligned} F_1(\pi\pi_\tau) &= \Gamma(\pi\pi_1\pi_\tau)\,\Gamma(\pi\pi_1)F(\pi^{-1}) \\ &= \Gamma(\pi\pi_1\pi_\tau)\,\Gamma(\pi\pi_1)F(\pi^{-1}\pi_\tau) \\ &= F_1(\pi). \end{aligned}$$

Hence, together with $F(\pi)$ the function $F_\pi(\pi) = T_{\pi_\tau}(s)F(\pi)$ also belongs to H^+. And so the invariance of the subspace H^+ is proved.

6. The Quasiregular Representation of G and its Decomposition into Irreducible Representations. *Quasiregular representation* of G is the name we use for the representation in the space of functions $f(x_1, x_2)$, $x_1, x_2 \in K$, for which

$$(f,f) = \int |f(x_1, x_2)|^2 \, dx_1 \, dx_2 < \infty.$$

The representation operator $T(g)$ corresponding to the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is given by the formula

$$T(g)f(x_1, x_2) = f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2). \tag{1}$$

It is obvious that $T(g_1 g_2) = T(g_1)T(g_2)$ for arbitrary g_1 and g_2 from G and that

$$(T(g)f, T(g)f) = (f, f).$$

So the operators $T(g)$ form a unitary representation of G.

We now obtain the decomposition of $T(g)$ into irreducible representations of the principal series.

We call a function $f(x_1, x_2)$ *homogeneous* of degree π, where π is a multiplicative character on $\mathbf{K}$, if it satisfies the condition

$$f(tx_1, tx_2) = \pi(t)\ |t|^{-1} f(x_1, x_2) \tag{2}$$

for every $t \neq 0$.

Every function $f(x_1, x_2)$ can be decomposed into homogeneous functions: we assign to every multiplicative character π the function

$$f_\pi(x_1, x_2) = \int f(tx_1, tx_2)\pi^{-1}(t)\ dt. \tag{3}$$

Clearly, $f_\pi(x_1, x_2)$ is a homogeneous function of degree π.

By the formula for the inverse Mellin transform we have

$$f(tx_1, tx_2) = |t|^{-1} \int f_\pi(x_1, x_2)\pi(t)\ d\pi, \tag{4}$$

where $d\pi$ is a suitably normed invariant measure on the character group. From (4) we obtain for $t = 1$ the required expansion of $f(x_1, x_2)$ into homogeneous functions

$$f(x_1, x_2) = \int f_\pi(x_1, x_2)\ d\pi. \tag{5}$$

The functions $f_\pi(x_1, x_2)$ being homogeneous are uniquely determined by their values on the line $x_2 = 1$. We set

$$\varphi_\pi(x) = f_\pi(x, 1). \tag{6}$$

We show now that the following Plancherel formula holds:

$$\int |f(x_1, x_2)|^2\ dx_1\ dx_2 = \int |\varphi_\pi(x)|^2\ dx\ d\pi. \tag{7}$$

For according to the Plancherel formula for the Mellin transform we have by (4)

$$\int |f(tx_1, tx_2)|^2\ |t|\ dt = \int |f_\pi(x_1, x_2)|^2\ d\pi.$$

Substituting $x_2 = 1$ we find

$$\int |f(tx, t)|^2 \, |t| \, dt = \int |\varphi_\pi(x)|^2 \, d\pi. \tag{8}$$

Integrating both sides of (8) with respect to x we obtain the Plancherel formula (7).

Let us see how the operator $T(g)$ acts on the function $\varphi_\pi(x)$. We have

$$\begin{aligned} T(g)\,\varphi_\pi(x) &= T(g) f_\pi(x_1, x_2)\big|_{\substack{x_1 = x \\ x_2 = 1}} \\ &= f_\pi(\alpha x + \gamma, \beta x + \delta) \\ &= f_\pi\left(\frac{\alpha x + \gamma}{\beta x + \delta}, 1\right) \pi(\beta x + \delta)\, |\beta x + \delta|^{-1}. \end{aligned}$$

Thus,

$$T(g)\,\varphi_\pi(x) = \varphi_\pi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) \pi(\beta x + \delta)\, |\beta x + \delta|^{-1}.$$

So we see that the functions $\varphi_\pi(x)$ are transformed according to the representation of the continuous series corresponding to the character π.

Hence, the formulae (5) and (7) give us the decomposition of the quasiregular representation of G into irreducible unitary representations of the continuous series.

7. The Supplementary Series of Irreducible Unitary Representations of G. Here we give a description of yet another series of irreducible unitary representations of G. The series is defined by analogy to the case of the field of complex or the field of real numbers (see Gel'fand et al. [27]).

Each representation of this series is given by a real number $\rho \neq 0$ in the interval $-1 < \rho < 1$, and is constructed in the space of functions $\varphi(x)$ on $\mathbf{K}$ for which

$$(\varphi, \varphi) = \frac{1}{\Gamma(\pi_\rho)} \int |x_1 - x_2|^{\rho - 1} \, \varphi(x_1) \overline{\varphi(x_2)} \, dx_1 \, dx_2 < \infty.$$

Here π_ρ denotes the character $\pi_\rho(x) = |x|^\rho$,

$$\Gamma(\pi_\rho) = \int \chi(x) \, |x|^{\rho - 1} \, dx. \tag{1}$$

The representation operator $T_\rho(g)$ corresponding to the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is given by the following formula:

$$T_\rho(g)\,\varphi(x) = \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) |\beta x + \delta|^{-\rho - 1}. \tag{2}$$

An immediate verification shows that

$$T_\rho(g_1g_2) = T_\rho(g_1)T_\rho(g_2)$$

for arbitrary matrices g_1 and g_2 from G and that

$$(T_\rho(g)\varphi, T_\rho(g)\varphi) = (\varphi, \varphi).$$

So the operators $T_\rho(g)$ provide a unitary representation of G.

We call the series of representations so constructed the *supplementary series.*

It can be shown that the representations $T_\rho(g)$ and $T_{-\rho}(g)$ are equivalent (for a proof in the case of the field of real or of complex numbers see [28]; for disconnected fields, the proof is similar). Therefore we may henceforth assume that $0 < \rho < 1$.

Another realization of the representations of the supplementary series is obtained by passing from the functions $\varphi(x)$ to their Fourier transforms

$$\tilde{\varphi}(u) = \int \varphi(x)\chi(-ux)\,dx.$$

In this realization the formula for the operator $T_\rho(g)$ takes the following from:

$$T_\rho(g)\varphi(u) = \int K_\rho(g \mid u, v)\,\varphi(v)\,dv, \tag{3}$$

where

$$K_\rho(g \mid u, v) = \int \chi\left(-ux + v\,\frac{\alpha x + \gamma}{\beta x + \delta}\right) |\beta x + \delta|^{-\rho-1}\,dx. \tag{4}$$

The expression for the kernel $K_\rho(g \mid u, v)$ can be written in a somewhat different form. Namely,

$$K_\rho(g \mid u, v) = |\beta|^{-1}\,\chi\left(\frac{\delta u + \alpha v}{\beta}\right)\int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)|t|^{-\rho-1}\,dt,$$

when $\beta \neq 0$, and

$$K_\rho(g \mid u, v) = |\delta|^{-\rho+1}\,\chi(\delta\gamma u)\,\delta(\delta^2 u - v),$$

when $\beta = 0$.

Let us find the expression for the scalar product (φ, φ) in the new realization. We observe that the Fourier transform carries the convolution of functions into their product. Since the Fourier transform of $|x|^{\rho-1}$ is

$$|x|^{\rho-1} = \Gamma(\pi_\rho)\,|u|^{-\rho},$$

where $\pi_\rho(x) = |x|^\rho$, we have

$$(\varphi, \varphi) = \int |u|^{-\rho}\,|\varphi(u)|^2\,du. \tag{5}$$

So the representation $T_\rho(g)$ of the supplementary series $(0 < \rho < 1)$ may be realized in the space of functions $\varphi(u)$ on $\mathbf{K}$ with the scalar product (5). The representation operator in this realization is given by the formulae (3) and (4).

All the representations of the supplementary series are irreducible.

The proof of this proposition is word for word the same as in the case of the principal continuous series (see **§ 3.4**).

8. The Special Representation of G**.** In **§ 3.7** we constructed the supplementary series of irreducible unitary representations $T_\rho(g)$, where $0 < \rho < 1$. Let us see what representations arise in the limiting case when $\rho = 0$ or $\rho = 1$.

Obviously, for $\rho = 0$ we obtain the representation of the principal continuous series corresponding to the character $\pi \equiv 1$.

We shall see presently that for $\rho = 1$ a new representation of G arises.

So we examine the representation

$$T_1(g)\,\varphi(x) = \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) |\beta x + \delta|^{-2}. \tag{1}$$

Let us clarify how we must define the space of functions $\varphi(x)$ so that the operators $T_1(g)$ are unitary operators in this space.

We note that the formula for the scalar product

$$(\varphi, \varphi) = \frac{1}{\Gamma(\pi_\rho)} \int |x_1 - x_2|^{\rho-1}\, \varphi(x_1)\overline{\varphi(x_2)}\, dx_1\, dx_2 \tag{2}$$

is meaningless for $\rho = 1$, because $\Gamma(\pi_\rho)\big|_{\rho=1} = 0$. Therefore we impose on $\varphi(x)$ the additional condition

$$\int \varphi(x)\, dx = 0. \tag{3}$$

For such functions we have

$$\int |x_1 - x_2|^{\rho-1}\, \varphi(x_1)\overline{\varphi(x_2)}\, dx_1\, dx_2|_{\rho=1} = 0,$$

and the expression (2) tends, as $\rho \to 1$, to the finite limit†

$$(\varphi, \varphi) = c\int \ln|x_1 - x_2|\; \varphi(x_1)\overline{\varphi(x_2)}\, dx_1\, dx_2. \tag{4}$$

We show now that the functions $\varphi(x)$ satisfying condition (3) form an invariant space under the operators $T_1(g)$. For we have

$$\int T_1(g)\,\varphi(x)\, dx = \int \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) |\beta x + \delta|^{-2}\, dx = \int \varphi(x)\, dx.$$

† $\ln|x|$ is the adjoint homogeneous function of degree of homogeneity $\pi \equiv 1$.

Consequently, if $\int \varphi(x)\,dx = 0$, then also

$$\int T_1(g)\,\varphi(x)\,dx = 0.$$

So we have obtained a representation in the space of functions $\varphi(x)$ for which

$$\int \varphi(x)\,dx = 0,$$

$$(\varphi, \varphi) = \int \ln|x_1 - x_2|\,\varphi(x_1)\overline{\varphi(x_2)}\,dx_1\,dx_2 < \infty.$$

The representation operator $T_1(g)$ is given by the following formula:

$$T_1(g)\,\varphi(x) = \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)|\beta x + \delta|^{-2}.$$

We call this representation $T_1(g)$ the *singular representation.*†

If we go over from the functions $\varphi(x)$ to their Fourier transforms $\tilde{\varphi}(u) = \int \varphi(x)\chi(-ux)\,dx$, we obtain another realization of the singular representation, in which it is constructed in the space of functions $\varphi(u)$ with

$$(\varphi, \varphi) = \int |u|^{-1}\,|\varphi(u)|^2\,du < \infty$$

(and so $\varphi(0) = 0$). The representation operator $T_1(g)$ has the form

$$T_1(g)\,\varphi(u) = \int K_1(g \mid u, v)\,\varphi(v)\,dv,$$

where

$$K_1(g \mid u, v) = \int \chi\left(-ux + v\,\frac{\alpha x + \gamma}{\beta x + \delta}\right)|\beta x + \delta|^{-2}\,dx.$$

9. Representations in the Spaces $\mathscr{D}_\pi$. In this subsection we give a brief description of nonunitary representations‡ of G.

With every multiplicative character $\pi(x)$ of $\mathbf{K}$ (here we do not require that $|\pi(x)| \equiv 1$) we associate a function space $\mathscr{D}_\pi$. This space consists of the functions $f(x_1, x_2)$, $x_1, x_2 \in \mathbf{K}$, that satisfy the following two requirements:

1. For a connected field $\mathbf{K}$ the functions $f(x_1, x_2)$ are continuous and infinitely differentiable everywhere except at $(0, 0)$. If $\mathbf{K}$ is disconnected, then for a matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ sufficiently close to the unit matrix

$$f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2) = f(x_1, x_2). \tag{1}$$

† In the case of a connected field the representation $T_1(g)$ is one of the representations of the continuous or the discrete series. Therefore the term "singular representation" refers only to disconnected fields.

‡ For details on these representations in the case of a connected field $\mathbf{K}$ see [28].

2. The functions $f(x_1, x_2)$ are homogeneous of weight π, that is,

$$f(tx_1, tx_2) = \pi(t)\,|t|^{-1} f(x_1, x_2) \tag{2}$$

for every $t \neq 0$.

There is a natural way of introducing a topology in $\mathscr{D}_\pi$ under which it becomes a complete space.

Now we give a representation of G in $\mathscr{D}_\pi$. If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then we define the corresponding representation operator $T_\pi(g)$ by the formula

$$T_\pi(g) f(x_1, x_2) = f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2). \tag{3}$$

The question of the irreducibility and equivalence of the representations $T_\pi(g)$ arises.† The problem for the field of complex numbers and the field of real numbers is discussed in detail in [28]. Here we state, without proof, the analogous results for a disconnected field.

We defined the *singular points* in the group of multiplicative characters π as the characters $\pi(x) = |x|$ and $\pi(x) = |x|^{-1}$.

1. Two representations $T_{\pi_1}(g)$ and $T_{\pi_2}(g)$, where π_1 is a nonsingular point, are equivalent if and only if $\pi_1 = \pi_2$ or $\pi_1 = \pi_2^{-1}$.

2. For nonsingular points π the representations $T_\pi(g)$ are irreducible, except when $\pi(x) = \operatorname{sign}_\tau x$. In this case $\pi(x) = \operatorname{sign}_\tau x$, $T_\pi(g)$ splits into the direct sum of two irreducible representations.

3. Let $\pi(x) = |x|$. Then the space $\mathscr{D}_\pi$ contains a one-dimensional invariant subspace $\mathscr{G}_\pi$. It consists of the functions $f(x_1, x_2) = \text{const}$. The space $\mathscr{D}_{\pi^{-1}}$ also contains an invariant subspace $\mathscr{F}_{\pi^{-1}}$ consisting of the functions $f(x_1, x_2)$ for which

$$\int f(x, 1)\, dx = 0.$$

Clearly, the factor space $\mathscr{D}_{\pi^{-1}}/\mathscr{F}_{\pi^{-1}}$ is one-dimensional and consequently, $\mathscr{D}_{\pi^{-1}}/\mathscr{F}_{\pi^{-1}} \cong \mathscr{G}_\pi$. It can be shown that $\mathscr{D}_\pi/\mathscr{G}_\pi \cong \mathscr{F}_{\pi^{-1}}$ that is, the representation in $\mathscr{F}_{\pi^{-1}}$ is equivalent to that in the factor space $\mathscr{D}_\pi/\mathscr{G}_\pi$.

Now let us find out for what π we can introduce in $\mathscr{D}_\pi$ a scalar product invariant under the representation operators. When this is possible, we can complete $\mathscr{D}_\pi$ relative to this scalar product and obtain a unitary representation of G. For connected fields the problem was investigated in [28]. Here, without proof, are the analogous results for disconnected fields.

An invariant scalar product exists in $\mathscr{D}_\pi$ if and only if one of the following conditions is satisfied:

† For the definitions of irreducibility and equivalence in the spaces see $\mathscr{D}_\pi$ [28].

1. $|\pi(x)| \equiv 1$; the corresponding unitary representation of G is the representation of the principal continuous series discussed in **§ 3.1.**

2. $\pi(x) = |x|^\rho$, where ρ is a real number, $0 < |\rho| < 1$; the corresponding unitary representations of G are the representations of the supplementary series discussed in **§ 3.7.**

Furthermore, for $\pi(x) = |x|^{-1}$ an invariant scalar product exists in the subspace $\mathscr{F}_\pi$ of functions from $\mathscr{D}_\pi$ satisfying the condition

$$\int f(x, 1)\, dx = 0;$$

the corresponding unitary representation of G is the singular representation discussed in **§ 3.8.**

In the classification of all irreducible representations of G, we mention one essential difference between the case of a connected field and a disconnected field **K**. In the case of a connected field it is sufficient to consider the space $\mathscr{D}_\pi$ and all its invariant subspaces and factor spaces, if $\mathscr{D}_\pi$ is reducible. It can be shown that in this way we obtain, to within equivalence, all the irreducible representations of G [27]. For a disconnected field **K** this is not so: the representations of the discrete series, which will be constructed in **§ 4**, are inequivalent in the spaces $\mathscr{D}_\pi$.

10. Spherical Functions. We say that an irreducible representation of G is of class I if the representation space contains a vector that is invariant under the subgroup U of integral matrices, that is, matrices whose elements are all $\mathfrak{p}$-adic integers.

Let us find that representations of the continuous series that belong to class I. As we know, a representation of the continuous series $T_\pi(g)$ can be realized in the space of functions $f(x) = f(x_1, x_2)$ satisfying the condition of homogeneity

$$f(tx) = \pi(t)\ |t|^{-1} f(x)$$

for every $t \neq 0$.

In this space we look for a function invariant under the operators $T(u)$ with $u \in U$.

We define the norm $|x|$ of a vector $x = (x_1, x_2)$ as the maximum of the norms of its coordinates:

$$|x| = \max\,(|x_1|, |x_2|). \tag{1}$$

It is easy to verify that any vector x' can be carried into another x'' of equal norm by a transformation from U. Hence it follows immediately that every function invariant under the compact subgroup U is of the form

$$f = F(|x|).$$

From the condition of homogeneity we obtain that

$$f = C\pi(|x|)\ |x|^{-1} = C\ |x|^{s-1}.$$

From this we conclude: the irreducible representations of the continuous series having a vector invariant under the subgroup U of integral matrices are precisely those that correspond to the character

$$\pi(x) = |x|^s.$$

This vector is uniquely determined to within a constant factor and has the following form:

$$f_0 = \sqrt{\frac{q}{1+q}}\, |x|^{s-1},$$

where $|x|$ is the norm of the vector $x = (x_1, x_2)$ defined by (1). The factor $\sqrt{\frac{q}{1+q}}$ is adjusted so that $\|f_0\| = 1$.

We define an elementary spherical function on G corresponding to an irreducible representation of class I as a function $\varphi(g)$ on G determined by the following formula:

$$\varphi(g) = (T(g)f_0, f_0)$$

where f_0 is a vector in the representation space that is invariant under U and such that $\|f_0\| = 1$, and the parentheses denote the scalar product. From the definition it follows immediately that the function $\varphi(g)$ is constant on the double cosets of U, that is,

$$\varphi(u_1 g u_2) = \varphi(g) \qquad \text{for arbitrary } u_1, u_2 \in U.$$

It can be shown that every matrix $g \in G$ can be represented in the following form:

$$g = u_1 \delta u_2,$$

where $u_1, u_2 \in U$ and δ is a diagonal matrix of the form

$$\delta = \begin{pmatrix} \mathfrak{p}^{-n} & 0 \\ 0 & \mathfrak{p}^{n} \end{pmatrix}, \qquad n \geq 0.$$

Thus, a spherical function $\varphi(g)$ is completely determined by its values on the matrices δ.

Let us compute $\varphi(\delta)$. Suppose, for the sake of definiteness, that $T(g)$ is a representation of the principal series, that is, $s = i\rho$ is a purely imaginary number. Then the scalar product is given by the following formula:

$$(f_1, f_2) = \int f_1(t, 1)\overline{f_2(t, 1)}\, dt.$$

So we have

$$\varphi(\delta) = \frac{q}{q+1} \int [\max (q^n\, |t|, q^{-n})]^{s-1} [\max (|t|, 1)]^{-s-1}\, dt.$$

We split this integral into three parts—the integral over the domain $|t| \leq q^{-2n}$, the integral over the domain $q^{-2n} < |t| \leq 1$, and the integral over the domain $|t| > 1$. Then we have

$$\frac{q}{q+1}\varphi(\delta)$$

$$= q^{-n(s-1)} \int\limits_{|t| \leq q^{-2n}} dt + q^{n(s-1)} \int\limits_{q^{-2n} < |t| \leq 1} |t|^{s-1}\, dt + q^{n(s-1)} \int\limits_{|t|>1} |t|^{-2}\, dt.$$

All the integrals in this expression are easily computed. Namely,

$$\int\limits_{|t| \leq q^{-2n}} dt = q^{-2n}; \quad \int\limits_{|t|>1} |t|^{-2}\, dt = q^{-1},$$

$$\int\limits_{q^{-2n} < |t| \leq 1} |t|^{s-1}\, dt = (1 - q^{-1})(1 + q^{-s} + q^{-2s} + \cdots + q^{-(2n-1)s})$$

$$= (1 - q^{-1})\frac{1 - q^{-2ns}}{1 - q^{-s}}.$$

As a result we find

$$\frac{q+1}{q}\varphi(\delta) = q^{-ns-n} + (1 - q^{-1})q^{-n}\frac{q^{ns} - q^{-ns}}{1 - q^{-s}} + q^{ns-n-1}.$$

After elementary transformations we obtain the following final form for a spherical function:

$$\varphi(\delta) = q^{-n}\frac{q^{1/2}(q^{(n+1/2)s} - q^{-(n+1/2)s}) - q^{-1/2}(q^{(n-1/2)s} - q^{-(n-1/2)s})}{(q^{s/2} - q^{-s/2})(q^{1/2} + q^{-1/2})}.$$

11. The Operator of the Horospherical Automorphism. Following Chapter 1 we define the horospherical subgroups of G as the subgroup Z of matrices of the form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ and all subgroups conjugate to Z. Horospheres in a homogeneous space X relative to G are orbits of horospherical subgroups. Thus, every horosphere on X consists of the points of the form

$$x_z = x_0 g_1 z g_2, \tag{1}$$

where x_0 is a fixed point in X, g_1 and g_2 fixed elements of G, and z ranges over Z.

From the definition it follows that every transitive family of horospheres on X either coincides with the space of cosets $\Omega = Z \setminus G$, or is obtained from Ω by an additional identification of points. We

call Ω the *space of horospheres*. This space Ω is isomorphic to a two-dimensional affine space over $\mathbf{K}$ from which the origin has been deleted.

Let us find all the horospheres in Ω. We give them by formula (1), where x_0 is the point of Ω corresponding to the unit class. We consider the matrix

$$g_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

in (1). Let us show that *if* $\beta = 0$, *then the horosphere* (1) *degenerates to a point*. For in this case we have $g_1 z = z' g_1$, where $z' \in Z$. Consequently, $x_z = x_0 g_1 g_2$ for every z, because $x_0 z = x_0$.

Now let $\beta \neq 0$. Then g_1 may be represented in the form

$$g_1 = z_1 s\, \delta z_2.$$

where z_1, $z_2 \in Z$, δ is a diagonal matrix, and

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

Thus, the equation of the horosphere (1) takes the following form:

$$x_z = x_0 s z\, \delta g_2. \tag{3}$$

So we see that *nondegenerate horospheres in* Ω *form a homogeneous family*. For they are all obtained by group translations from $x_z = x_0 s z$.

When we go over in (3) to coordinates and bear in mind that $x_0 = (1, 0)$, we obtain the following equation of horospheres:

$$x_1 = \alpha z + \gamma, \quad x_2 = \beta z + \delta, \quad z \in \mathbf{K}, \quad \alpha\delta - \beta\gamma = 1. \tag{4}$$

Thus, *the horospheres in the space* Ω *of points* $x = (x_1, x_2)$, $x \neq 0$, *are all the lines that do not pass through the origin of coordinates.*

Let $\varphi(x)$ be a test function in Ω. We associate with it integrals of $\varphi(x)$ over all the horospheres (that is, lines) in Ω:

$$\psi(g) = \int_Z \varphi(x_0 s z g)\, dz. \tag{5}$$

Observe that $\psi(zg) = \psi(g)$ for every $z \in Z$. Thus, ψ may be regarded as a function in the space $\Omega = Z \setminus G$, and we can write $\psi(x)$ instead of $\psi(g)$.

Hence, the map

$$B \colon \varphi(x) \rightarrow \psi(x) \tag{6}$$

carries functions on Ω again into functions on Ω. We call this map B the *horospherical automorphism.*

In coordinate notation the operator B is given, as is easy to see, by the following formula:

$$B\varphi(x_1, x_2) = \int_K \varphi(x_1 z + y_1, x_2 z + y_2)\, dz. \tag{7}$$

where y_1 and y_2 are arbitrary elements from $\mathbf{K}$ connected with x_1 and x_2 by the relation

$$x_1 y_2 - x_2 y_1 = 1.$$

For example, when $x_1 \neq 0$, (7) can be written in the form

$$B\varphi(x_1, x_2) = \int_K \varphi(x_1 z, x_2 z + x_1^{-1})\, dz. \tag{8}$$

The fundamental properties of B are the following:

1. *The operator B commutes with the operators of group translation* $f(x) \to f(xg)$.

This follows immediately from (5).

2. *The operator B carries homogeneous functions of weight π into homogeneous functions*† *of weight π^{-1}.*

This follows immediately from (8).

From property 2 of B it follows that *the operator B^2 carries every space $\mathscr{D}_\pi$ of homogeneous functions into itself.* Since $\mathscr{D}_\pi$ is irreducible and B^2 commutes with the representation operators, this operator is a multiple of the unit operator on each space $\mathscr{D}_\pi$.

$$B^2 \varphi_\pi = \lambda(\pi)\, \varphi_\pi$$

for every function $\varphi_\pi \in \mathscr{D}_\pi$.

Our main task is to compute the factor of proportionality $\lambda(\pi)$. In this subsection we find $\lambda(\pi)$ for the field of real numbers and for a disconnected field; see formulae (15) and (30).

We introduce two homogeneous function of weight π. To construct them we extend the character π to a multiplicative character on the quadratic extension $\mathbf{K}(\sqrt{\varepsilon})$ of $\mathbf{K}$. We denote the character so obtained, as before, by π. We set

$$\varphi_\pi^{(1)}(x, y) = \pi(x + \sqrt{\varepsilon}\, y)\, |x + \sqrt{\varepsilon}\, y|^{-1},$$

$$\varphi_\pi^{(2)}(x, y) = \pi(x - \sqrt{\varepsilon}\, y)\, |x - \sqrt{\varepsilon}\, y|^{-1},$$

where $x, y \in \mathbf{K}$.

† That is, functions satisfying the relation

$$\varphi(tx) = \pi(t)\, |t|^{-1} \varphi(x), \qquad t \in \mathbf{K}$$

Let us compute $B\varphi_\pi^{(1)}$ and $B\varphi_\pi^{(2)}$.

To begin with we consider the field $\mathbf{K}$ of real numbers. Here we have $(\sqrt{\varepsilon} = i)$:

$$= \int_{-\infty}^{+\infty} \pi(xz + i(yz + x^{-1}))\ |xz + i(yz + x^{-1})|^{-1}\, dz.$$

We transform this integral. We have

$$xz + i(yz + x^{-1}) = (x + iy)\left(z + \frac{i + x^{-1}y}{x^2 + y^2}\right)$$

$$= (x - iy)^{-1}\left[(x^2 + y^2)\left(z + \frac{x^{-1}y}{x^2 + y^2}\right) + i\right].$$

Consequently, after a suitable change of variable we find

$$B\varphi_\pi^{(1)}(x,y) = \pi^{-1}(x - iy)\ |x - iy|^{-1} \int_{-\infty}^{+\infty} \pi(z + i)\ |z + i|^{-1}\, dz.$$

Thus,

$$B\varphi_\pi^{(1)}(x,y) = \lambda^{(1)}(\pi)\, \varphi_{\pi^{-1}}^{(2)}(x,y), \tag{9}$$

where

$$\lambda^{(1)}(\pi) = \int_{-\infty}^{+\infty} \pi(z + i)\ |z + i|^{-1}\, dz. \tag{10}$$

The integral (10) can be expressed directly in terms of the classical Beta-function. For let

$$\pi(x) = |x|^s \operatorname{sign}^\nu x, \qquad \nu = 0, 1.$$

We extend the character π to the field of complex numbers by the following formula:

$$\pi(z) = |z|^s e^{\nu \arg z}. \tag{11}$$

It is not hard to check that then

$$\lambda^{(1)}(\pi) = \begin{cases} B\left(-\frac{s}{2}, \frac{1}{2}\right), & \text{when } \nu = 0, \\ iB\left(-\frac{s-1}{2}, \frac{1}{2}\right), & \text{when } \nu = 1. \end{cases} \tag{12}$$

Similarly we find that

$$B\varphi_\pi^{(2)}(x,y) = \lambda^{(2)}(\pi)\, \varphi_{\pi^{-1}}^{(1)}(x,y), \tag{13}$$

where

$$\lambda^{(2)}(\pi) = (-1)^\nu \lambda^{(1)}(\pi).$$

From this formulac it follows that

$$B^2\varphi_\pi^{(1)} = \lambda(\pi)\varphi_\pi^{(1)}, \qquad B^2\varphi_\pi^{(2)} = \lambda(\pi)\varphi_\pi^{(2)}, \tag{14}$$

where

$$\lambda(\pi) = \lambda^{(1)}(\pi)\lambda^{(2)}(\pi^{-1}) = \begin{cases} -\dfrac{2\pi}{s}\cot\dfrac{\pi s}{2}, & \text{when } \nu = 0. \\ \dfrac{2\pi}{s}\tan\dfrac{\pi s}{2}, & \text{when } \nu = 1. \end{cases} \tag{15}$$

Now we consider a disconnected field $\mathbf{K}$. In this case we have

$$B\varphi_\pi^{(1)}(x,y) = \int_{\mathbf{K}} \pi(xz + \sqrt{\varepsilon}\,(yz + x^{-1}))\,|xz + \sqrt{\varepsilon}\,(yz + x^{-1})|^{-1}\,dz.$$

As in the case of the field of real numbers, we find

$$B\varphi_\pi^{(1)}(x,y) = \lambda^{(1)}(\pi)\,\varphi_{\pi-1}^{(2)}(x,y), \tag{16}$$

where

$$\lambda^{(1)}(\pi) = \int_{\mathbf{K}} \pi(z + \sqrt{\varepsilon})\,|z + \sqrt{\varepsilon}|^{-1}\,dz. \tag{17}$$

Now we compute the integral (17).

The character $\pi(x)$ is given by the following formula:

$$\pi(x) = |x|^s\,\theta(x), \tag{18}$$

where s is a complex number and $\theta(\mathfrak{p}) = 1$.

We begin with $\theta(x) \equiv 1$. Then we have

$$\lambda^{(1)}(\pi) = \int_{|x|>1} |x|^{s-1}\,dx + \int_{|x|\le 1} dx = \frac{1 - q^{s-1}}{1 - q^s}. \tag{19}$$

Now we consider the case $\theta \not\equiv 1$. Let n be the rank of θ in $\mathbf{K}$.

We recall that the rank is the smallest natural number n for which

$$\theta(1 + \mathfrak{p}^n x) \equiv 1, \qquad |x| \le 1\,.$$

The rank of θ is the same, whether regarded as a character on $\mathbf{K}$ or on $\mathbf{K}(\sqrt{\varepsilon})$.

First we show that

For we have

$$I_k = q^{sk} \int\limits_{|t|=1} \theta(\mathfrak{p}^{-k}t + \sqrt{\varepsilon})\, dt = p^{sk} \int\limits_{|t|=1} \theta(t + \mathfrak{p}^k\sqrt{\varepsilon})\, dt.$$

Hence it follows that for every $x \in \mathbf{K}$, $|x| \leq 1$,

$$I_k\theta(1 + \mathfrak{p}^{n-1}x) = q^{sk} \int\limits_{|t|=1} \theta(t(1 + \mathfrak{p}^{n-1}x) + \mathfrak{p}^k\sqrt{\varepsilon})\, dt$$

$$= q^{sk} \int\limits_{|t|=1} \theta(t + \mathfrak{p}^k\sqrt{\varepsilon}) = I_k.$$

But $\theta(1 + \mathfrak{p}^{n-1}x) \not\equiv 1$. Consequently, $I_k = 0$. By what we have proved we obtain the following expression for $\lambda^{(1)}(\pi)$:

$$\lambda^{(1)}(\pi) = \int\limits_{|z|\leq 1} \theta(z + \sqrt{\varepsilon})\, dz = q^{-n} \sum_{z\in O/\mathfrak{p}^n O} \theta(z + \sqrt{\varepsilon}). \qquad (20)$$

On the basis of this formula we can show that

$$|\lambda^{(1)}(\pi)|^2 = q^{-n}. \qquad (21)$$

Proof. We have

$$|\lambda^{(1)}(\pi)|^2 = q^{-2n} \sum_{z,u\in O/\mathfrak{p}^n O} \theta\left(\frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right).$$

(z and u range over a set consisting of one representative from each coset of $O/\mathfrak{p}^nO$.) In this sum we separate the term with $z = u$. We find

$$|\lambda^{(1)}(\pi)|^2 = q^{-n} + q^{-2n} \sum_{z\neq u} \theta\left(\frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right).$$

We show now that the second term is zero. For this purpose we consider the set of values mod $\mathfrak{p}^n$ of $\dfrac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}$. It is not hard to verify that this set is preserved under multiplication by elements of the form $x_s = 1 + \mathfrak{p}^{n-1}s$, $|s| \leq 1$. Hence,

$$\sum_{z\neq u} \theta\left(\frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right) = \sum_{z\neq u} \theta\left(x_s \frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right) = \pi(x_s) \sum_{z\neq u} \theta\left(\frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right).$$

Since $\theta(x_s) \not\equiv 1$, it follows immediately that

$$\sum_{z\neq u} \theta\left(\frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right) = 0.$$

We give another derivation of the formula (21) based on results of **§ 2.6**.

We introduce the Gamma-function $\Gamma_\varepsilon(\pi)$ in $\mathbf{K}(\sqrt{\varepsilon})$ by the following formula:

$$\Gamma_\varepsilon(\pi) = \int\limits_{K(\sqrt{\varepsilon})} \chi\left(\frac{t - \bar{t}}{2\sqrt{\varepsilon}}\right) \pi(t)\, d^*t, \qquad (22)$$

and show that

$$\lambda^{(1)}(\pi) = \frac{\Gamma_\varepsilon(\pi\pi_0)}{\Gamma(\pi\pi_0)}, \tag{23}$$

where $\pi_0(x) = |x|$.

By the substitution $t = xy + \sqrt{\varepsilon}\, y$, where $x, y \in \mathbf{K}$, the integral (22) reduces to the form:

$$\Gamma_\varepsilon(\pi) = \int \chi(y)\pi(y)\, |y|^{-1}\, dy \cdot \int \pi(x + \sqrt{\varepsilon})\, |x + \sqrt{\varepsilon}|^{-2}\, dx.$$

Equation (23) follows immediately from this.

To obtain (21) from (23) we use the following formula, which was derived in **§ 2.6**:

$$|\Gamma(\pi)| = q^{n(\mathrm{Re}\, s - \frac{1}{2})}. \tag{24}$$

We show that

$$|\Gamma_\varepsilon(\pi)| = q^{n(\mathrm{Re}\, s-1)}. \tag{24'}$$

Indeed in passing from $\mathbf{K}$ to $\mathbf{K}(\sqrt{\varepsilon})$ the number q (the order of the residue class field O/P) is replaced by q^2, and the rank of the character π is preserved. Hence, by (24), we have $|\Gamma_\varepsilon(\pi)| = q^{2n(\mathrm{Re}\, s/2-1/2)} = q^{n(\mathrm{Re}\, s-1)}$.

(21) is an immediate consequence of (23), (24), and (24').

We have now obtained the final formula for the operator B: If $\pi(x) = |x|^s\, \theta(x)$, $\theta(\mathfrak{p}) = 1$, then

$$B\varphi_\pi^{(1)}(x,y) = \lambda^{(1)}(\pi)\, \varphi_{\pi^{-1}}^{(2)}(x,y), \tag{25}$$

where

$$\lambda^{(1)}(\pi) = \begin{cases} \dfrac{1 - q^{s-1}}{1 - q^s}, & \text{when } \theta(x) \equiv 1, \\ q^{-n/2}\mu^{(1)}(\pi), & \text{when } \theta(x) \not\equiv 1. \end{cases} \tag{26}$$

Here n is the rank of θ, and

$$|\mu^{(1)}(\pi)| = 1.$$

The computation of $\mu^{(1)}(\pi)$ is a task of considerably greater complexity. However, for our purposes the value of $\mu^{(1)}(\pi)$ is not required.

Similarly, we have

$$B\varphi_\pi^{(2)}(x,y) = \lambda^{(2)}(\pi)\, \varphi_{\pi^{-1}}^{(1)}(x,y). \tag{27}$$

Next we show that the functions $\lambda^{(2)}(\pi)$ and $\lambda^{(1)}(\pi)$ are connected by the following relation:

$$\lambda^{(1)}(\pi)\, \lambda^{(2)}(\pi^{-1}) = \pi(-1) q^{-n}. \tag{28}$$

For by analogy with (20) we have

$$\lambda^{(2)}(\pi^{-1}) = \int_{|z| \le 1} \theta^{-1}(z - \sqrt{\varepsilon})\, dz$$

$$= \theta(-1) \int_{|z| \le 1} \theta^{-1}(z + \sqrt{\varepsilon})\, dz = \theta(-1)\, \overline{\lambda^{(1)}(\pi)}.$$

Consequently,

$$\lambda^{(1)}(\pi)\lambda^{(2)}(\pi^{-1}) = \theta(-1)\,|\lambda^{(1)}(\pi)|^2 = \theta(-1)q^{-n}.$$

On the basis of these results we have

$$B^2\varphi_\pi^{(1)} = \lambda(\pi)\varphi_\pi^{(1)}, \qquad B^2\varphi_\pi^{(2)} = \lambda(\pi)\varphi_\pi^{(2)}, \tag{29}$$

where

$$\lambda(\pi) = \lambda^{(1)}(\pi)\lambda^{(2)}(\pi^{-1}) = \begin{cases} \dfrac{(1-q^{s-1})(1-q^{-s-1})}{(1-q^s)(1-q^{-s})}, \text{ when } \theta \equiv 1, \\ q^{-n}\pi(-1), \text{ when the rank of } \theta \text{ is } n \neq 0, \end{cases} \tag{30}$$

that is, in accordance with **§ 2.6**

$$\lambda(\pi) = \Gamma(\pi)\Gamma(\pi^{-1}). \tag{31}$$

§ 4. THE DISCRETE SERIES OF IRREDUCIBLE UNITARY REPRESENTATIONS OF G

1. Description of the Representations of the Discrete Series. Here we show that with every quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$, a certain discrete series of irreducible unitary representations of G is connected. Thus, for the field of real numbers there is one discrete series, and for a disconnected field there are three discrete series of irreducible unitary representations of G.

As a preliminary we recall the formula for the operators of the continuous series in the χ-realization. If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then the corresponding representation operator $T_\pi(g)$ is given by the formula

$$T_\pi(g)\varphi(u) = \int K_\pi(g \mid u, v)\varphi(v)\,dv,$$

where

$$K_\pi(g \mid u, v) = |\beta|^{-1}\chi\left(\frac{\delta u + \alpha v}{\beta}\right)\int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\,d^*t,$$

when $\beta \neq 0$;

$$K_\pi(g \mid u, v) = \pi(\delta)\,|\delta|\,\chi(\delta\gamma u)\delta(\delta^2 u - v),$$

when $\beta = 0$.

Here π is the multiplicative character on $\mathbf{K}$ that generates the representation.

We define the representations of the discrete series by similar formulae.

Let $\mathbf{K}(\sqrt{\tau})$ *be a quadratic extension of* $\mathbf{K}$, $\pi(t)$ *a multiplicative character on* $\mathbf{K}(\sqrt{\tau})$. *We consider the space* H *of functions* $\varphi(u)$ *on* $\mathbf{K}$ *for which*

$$(\varphi, \varphi) = \int |\varphi(u)|^2 \, du < \infty.$$

With every matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *we associate the operator* $T_\pi(g)$ *that is defined in* H *by the following formula:*

$$T_\pi(g)\,\varphi(u) = \int K_\pi(g \mid u, v)\,\varphi(v)\,dv, \tag{1}$$

where

$$K_\pi(g \mid u, v) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \operatorname{sign}_\tau u\, \chi\left(\frac{\delta u + \alpha v}{\beta}\right)$$

$$\int\limits_{t\bar{t}=vu^{-1}} \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\,d^*t, \tag{2}$$

when $\beta \neq 0$, $\operatorname{sign}_\tau u = \operatorname{sign}_\tau v$;

$$K_\pi(g \mid u, v) = 0, \tag{3}$$

when $\operatorname{sign}_\tau u \neq \operatorname{sign}_\tau v$;

$$K_\pi(g \mid u, v) = \operatorname{sign}_\tau \delta \cdot \pi(\delta)\,|\delta|\,\chi(\delta\gamma u)\,\delta(\delta^2 u - v), \tag{4}$$

when $\beta = 0$.

Here d^*t denotes the measure that is uniquely determined on the circle $t\bar{t} = vu^{-1}$ by the condition $d^*(tt_0) = d^*t$ for every t_0 with $t_0\bar{t}_0 = 1$; $\int d^*t = 1$; $a_\tau = 2(1 + q^{-1})(1 + |\tau|)^{-1}$. The coefficient c_τ is determined by the formula

$$c_\tau^{-1} = \int \chi(t\bar{t})\,dt, \tag{5}$$

where the integration is taken over the plane $\mathbf{K}(\sqrt{\tau})$. The precise meaning and the value of this integral were indicated in **§ 2.7**.

In **§ 4.3** and **§ 4.4** we shall show that *the operators* $T_\pi(g)$ *form a unitary representation of* G.

We make some preliminary remarks on the representations $T_\pi(g)$:

1. We see that the operators $T_\pi(g)$ are defined essentially by the same formulae as those for the representation operators of the continuous series. The only important difference is that the integration in (2) taken not over a "line," but over the circle

$t\bar{t} = vu^{-1}$ on the plane $\mathbf{K}(\sqrt{\tau})$. The points t of this circle are characterized by the condition that $ut + vt^{-1}$ must belong to $\mathbf{K}$.

2. In **§ 4.6** we shall show that if $\pi_1 = \pi_2$ on the circle $t\bar{t} = 1$, then the representations $T_{\pi_1}(g)$ and $T_{\pi_2}(g)$ are equivalent. Hence the representations $T_\pi(g)$ are, in fact, given by the characters on $t\bar{t} = 1$, and so the set of these representations is discrete. This is the explanation of the name "discrete series."

3. The representations $T_\pi(g)$ are reducible. For let H^+ be the subspace of functions $\varphi(u)$ for which $\varphi(u) = 0$ when $\operatorname{sign}_\tau u = -1$; H^- the subspace of functions for which $\varphi(u) = 0$ when $\operatorname{sign}_\tau u = 1$.

From the formulae for the representation operators it is immediately clear that H^+ and H^- are invariant subspaces.

From now on we denote the representations in H^+ and H^-, respectively, by $T_\pi^+(g)$ and $T_\pi^-(g)$. These representations are irreducible (see **§ 4.6**).

So we see that *every discrete series of irreducible unitary representations consists of two halves—the representations* $T_\pi^+(g)$ *and* $T_\pi^-(g)$. *The first are realized in the subspace of functions* $\varphi(u)$ *for which* $\varphi(u) = 0$ *when* $\operatorname{sign}_\tau u = -1$; *the second in the supplementary subspace.*

A similar series of representations arises in the case of a finite field $\mathbf{K}_q$. We assume that the characteristic of $\mathbf{K}_q$ is different from two. Then $\mathbf{K}_q$ has precisely one quadratic extension. The series of representations connected with this extension is realized on the functions $\varphi(u)$, where u ranges over the elements of $\mathbf{K}_q$ other than zero. The representation operator has the form

$$T_\pi(g)\varphi(u) = \sum_{v\neq 0} K_\pi(g \mid u, v)\varphi(v),$$

where

$$K_\pi(g \mid u, v) = -\chi\left(\frac{\delta u + \alpha v}{\beta}\right)\sum_{t\bar{t}=vu^{-1}} \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t),$$

when $\beta \neq 0$;

$$K_\pi(g \mid u, v) = \pi(\delta)\chi(\delta\gamma u)\delta(\delta^2 u - v),$$

when $\beta = 0$. Here $\delta(u)$ is the Delta-function: $\delta(u) = 0$ for $u \neq 0$, $\delta(0) = 1$.

In contrast to an infinite field, the representations $T_\pi(g)$ turn out to be irreducible (except when $\pi(x) = \pm 1$). It can be shown that $T_\pi(g)$ and $T_{\pi^{-1}}(g)$ are equivalent representations.

2. The Continuous Dependence of the Operators $T_\pi(g)$ on g.

The operators $T_\pi(g)$ corresponding to the matrices $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ were defined by different formulae in the cases $\beta \neq 0$ and $\beta = 0$.

We show now that *the formula for* $T_\pi(g)$ *in the special case* $\beta = 0$ *is obtained by a limit process from the formula for* $T_\pi(g)$ *corresponding to a matrix in general position.* In this way we establish that the operators $T_\pi(g)$ depend continuously on g.

As a preliminary we put the formulae for the operators into a somewhat different form.

By **§ 4.1** the operator $T_\pi(g)$, $\beta \neq 0$, is given by the following formula:

$$T_\pi(g)\,\varphi(u) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \operatorname{sign}_\tau u \int\limits_{t\bar t = vu^{-1}} \chi\left(\frac{\delta u + \alpha v}{\beta} - \frac{1}{\beta}(ut + vt^{-1})\right) \pi(t)\,\varphi(v)\, d^*t\, dv, \tag{1}$$

where the integration is with respect to t is taken over the circle $t\bar t = vu^{-1}$. When we substitute $v = ut\bar t$ in the integral, we may rewrite the formula as follows:

$$T_\pi(g)\,\varphi(u) = c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} |u| \operatorname{sign}_\tau u \int \chi\left(\frac{u}{\beta}(\delta + \alpha t\bar t - t - \bar t)\right) \pi(t)\,\varphi(ut\bar t)\, dt, \tag{2}$$

where the integration is taken over the whole plane $\mathbf{K}(\sqrt{\tau})$. We make the change of variable: $t = \beta t' + \delta$. After elementary transformations we find

$$T_\pi(g)\,\varphi(u) = c_\tau\, |\beta| \operatorname{sign}_\tau \beta |u| \operatorname{sign}_\tau u\, \chi(\delta\gamma u)$$
$$\int \chi[(\alpha\beta t\bar t + \gamma\beta(t + \bar t))u] \times \pi(\beta t + \delta)\,\varphi(u(\beta t + \delta)(\beta\bar t + \delta))\, dt. \tag{3}$$

Let us see what the limiting value of this expression is, as $\beta \to 0$. We assume that $\operatorname{sign}_\tau \beta$ remains constant. Let β_0 be a fixed element such that $\operatorname{sign}_\tau \beta_0 = \operatorname{sign}_\tau \beta$. Then we have

$$\beta = \beta_0 \sigma\bar\sigma,$$

where σ is an element from $\mathbf{K}(\sqrt{\tau})$. In the integral (3) we make the change of variable $t = \bar\sigma^{-1} t'$, and find

$$T_\pi(g)\,\varphi(u) = c_\tau\, |\beta_0| \operatorname{sign}_\tau \beta_0\, |u| \operatorname{sign}_\tau u \cdot \chi(\delta\gamma u)$$
$$\int \chi[u(\alpha\beta_0 t\bar t + \gamma\beta_0(\sigma t + \bar\sigma\bar t))]\,\pi(\beta_0\sigma t + \delta)\,\varphi(u(\beta_0\sigma t + \delta)(\beta_0\bar\sigma\bar t + \delta))\, dt. \tag{4}$$

We are interested in the limit of this expression, as $\sigma \to 0$. Let us perform a formal limit passage under the integral sign. Then we obtain

$$T_\pi(g)\,\varphi(u) = c_\tau\, |\beta_0| \operatorname{sign}_\tau \beta_0\, |u| \operatorname{sign}_\tau u \cdot \chi(\delta\gamma u)\,\pi(\delta)\,\varphi(\delta^2 u) \int \chi(u\alpha\beta_0 t\bar t)\, dt. \tag{5}$$

However,

$$\int \chi(u\alpha\beta_0 t\bar{t})\,dt = c_\tau^{-1}\frac{\operatorname{sign}_\tau(u\alpha\beta_0)}{|u\alpha\beta_0|} + \frac{a_\tau}{2}\,\delta(\alpha\beta_0 u)$$

(see **§ 2.7**). Substituting this expression in (5) and bearing in mind that in the limit matrix $\alpha = \delta^{-1}$, we find

$$T_\pi(g)\,\varphi(u) = \operatorname{sign}_\tau \delta \cdot \pi(\delta)\ |\delta|\ \chi(\delta\gamma u)\,\varphi(\delta^2 u),$$

that is,

$$T_\pi(g)\,\varphi(u) = \operatorname{sign}_\tau \delta \cdot \pi(\delta)\ |\delta|\ \chi(\delta\gamma u)\int \delta(\delta^2 u - v)\,\varphi(v)\,dv.$$

So we have obtained precisely the formula (4) of **§ 4.1** for the operator $T_\pi(g)$ corresponding to the matrix $g = \begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}$.

The limiting process, as $\sigma \to 0$, as we have carried it out, is not completely rigorous. To make the argument rigorous we have to introduce instead of the $T_\pi(g)$ auxiliary operators by adding the factor $|u(\alpha\beta_0 t\bar{t} + \gamma\beta_0(\sigma t + \bar{\sigma}\bar{t}))|^\lambda$ under the integral (4) (λ is a complex number). We split the integral so obtained into two terms: over $|t| < 1$ and over $|t| \geq 1$. It is easy to see that for each of these integrals there is a domain of values λ for which it converges absolutely and uniformly in σ, as $\sigma \to 0$; and then the limit passage, as $\sigma \to 0$, under the integral sign is possible. We do not wish to go into details of this common technique in generalized functions here.

3. Proof of the Relation $T_\pi(g_1 g_2) = T_\pi(g_1)\cdot T_\pi(g_2)$. We show that the operators $T_\pi(g)$ actually give a representation of G, that is,

$$T_\pi(g_1 g_2) = T_\pi(g_1)\,T_\pi(g_2) \tag{1}$$

for arbitrary matrices g_1 and g_2 from G.

The operators $T_\pi(g_1)$, $T_\pi(g_2)$, and $T_\pi(g_1 g_2)$ are given, respectively, by the kernels $K_\pi(g_1 \mid u, v)$, $K_\pi(g_2 \mid u, v)$, and $K_\pi(g_1 g_2 \mid u, v)$. So we must show that

$$\int K_\pi(g_1 \mid u, w) K_\pi(g_2 \mid w, v)\,dw = K_\pi(g_1 g_2 \mid u, v). \tag{2}$$

Let

$$g_1 = \begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix},\qquad g_2 = \begin{pmatrix}\alpha_2 & \beta_2\\ \gamma_2 & \delta_2\end{pmatrix},\qquad g_1 g_2 = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

It is sufficient to discuss the case $\beta_1 \neq 0$, $\beta_2 \neq 0$, $\beta \neq 0$. For the special cases when at least one of the elements β_1, β_2, β is equal to zero, the relation (1) can be obtained later by a limit process.

When we substitute in (2) the expressions for the kernels from **§ 4.1** we find

$$\mathscr{T} \equiv \int K_\pi(g_1 \mid u, w) K_\pi(g_2 \mid w, v)\, dw$$

$$= a_\tau^2 c_\tau^2 \operatorname{sign}_\tau (\beta_1\beta_2)\, |\beta_1\beta_2|^{-1}\, \chi\left(\frac{\delta_1 u}{\beta_1} + \frac{\alpha_2 v}{\beta_2}\right)$$

$$\int \int_{t\bar{t}=w/u} \int_{s\bar{s}=v/w} \chi\left(-\frac{u}{\beta_1}(t+\bar{t}) - \frac{w}{\beta_2}(s+\bar{s}) + \frac{\alpha_1 w}{\beta_1} + \frac{\delta_2 w}{\beta_2}\right) \pi(ts)\, d^*s\, d^*t\, dw. \tag{3}$$

We make the change of variable $s = t^{-1}\sigma$ and obtain

$$\mathscr{T} = a_\tau^2 c_\tau^2 \operatorname{sign}_\tau (\beta_1\beta_2)\, |\beta_1\beta_2|^{-1}\, \chi\left(\frac{\delta_1 u}{\beta_1} + \frac{\alpha_2 v}{\beta_2}\right)$$

$$\int_{\sigma\bar{\sigma}=vu} \int \int_{t\bar{t}=wu} \chi\left(-\frac{u}{\beta_1}(t+\bar{t}) - \frac{u}{\beta_2}(\sigma\bar{t}+\bar{\sigma}t) + \frac{\beta}{\beta_1\beta_2} w\right) \pi(\sigma)\, d^*t\, dw\, d^*\sigma. \tag{4}$$

We compute the inner integral

$$I = \int \int_{t\bar{t}=w/u} \chi\left(-\frac{u}{\beta_1}(t+\bar{t}) - \frac{u}{\beta_2}(\sigma\bar{t} + \bar{\sigma}t) + \frac{\beta}{\beta_1\beta_2} w\right) d^*t\, dw$$

separately.

Substituting $w = ut\bar{t}$ in the integral we may rewrite it as an integral over the plane $\mathbf{K}(\sqrt{\tau})$:

$$I = a_\tau^{-1}\, |u| \int \chi\left(-u(\bar{a}t + a\bar{t}) + u\frac{\beta}{\beta_1\beta_2} t\bar{t}\right) dt,$$

where $a = \dfrac{1}{\beta_1} + \dfrac{\sigma}{\beta_2}$.

We make the substitution $t = t' + \dfrac{\beta_1\beta_2}{\beta} a$ and obtain

$$I = a_\tau^{-1}\, |u|\, \chi\left(-u\frac{\beta_1\beta_2}{\beta} a\bar{a}\right) \int \chi\left(u\frac{\beta}{\beta_1\beta_2} t\bar{t}\right) dt.$$

However,

$$\int \chi\left(u\frac{\beta}{\beta_1\beta_2} t\bar{t}\right) dt = c_\tau^{-1} \operatorname{sign}_\tau \left(\frac{u\beta}{\beta_1\beta_2}\right) \left|\frac{u\beta}{\beta_1\beta_2}\right|^{-1} + \frac{a_\tau}{2}\, \delta\left(\frac{\beta}{\beta_1\beta_2} u\right).$$

Consequently,

$$I = a_\tau^{-1}c_\tau^{-1}\operatorname{sign}_\tau\left(\frac{u\beta}{\beta_1\beta_2}\right)\left|\frac{\beta_1\beta_2}{\beta}\right|\chi\left(-u\,\frac{\beta_1\beta_2}{\beta}\,a\bar{a}\right)$$

$$= a_\tau^{-1}c_\tau^{-1}\operatorname{sign}_\tau\left(\frac{u\beta}{\beta_1\beta_2}\right)\left|\frac{\beta_1\beta_2}{\beta}\right|\chi\left(-\frac{\beta_2}{\beta_1\beta_2}\,u-\frac{u}{\beta}\,(\sigma-\bar{\sigma})-\frac{\beta_1}{\beta_2\beta}\,v\right).$$

We substitute this expression in (4) and use the easily verified relations

$$\frac{\delta_1}{\beta_1}-\frac{\beta_2}{\beta_1\beta}=\frac{\delta}{\beta},\qquad \frac{\alpha_2}{\beta_2}-\frac{\beta}{\beta_2\beta}=\frac{\alpha}{\beta}.$$

So we find

$$\mathscr{T}=a_\tau c_\tau\,\frac{\operatorname{sign}_\tau\beta}{|\beta|}\operatorname{sign}_\tau u\cdot\chi\left(\frac{\delta u+\alpha v}{\beta}\right)\int\limits_{\sigma\bar{\sigma}=v/u}\chi\left(-\frac{u}{\beta}\,(\sigma+\bar{\sigma})\right)\pi(\sigma)\,d^*\sigma$$

$$= K_\pi(g_1g_2\mid u,v).$$

The relation (2) is now proved.

There is a certain lack of rigor in our arguments, because we have computed the integral (2), which diverges in the usual sense. This can be avoided by considering instead of $K_\pi(g\mid u,v)$ the auxiliary kernels

$$K_\pi(g\mid u,v\mid\lambda)=K_\pi(g\mid u,v)\,|v|^\lambda,$$

where λ is a complex number. We form the integral

$$\mathscr{T}_\lambda=\int K_\pi(g_1\mid u,w\mid\lambda)K_\pi(g_2\mid w,v\mid\lambda)\,dw.$$

We split it into two integrals—over $|w|<1$ and over $|w|\geq 1$. It is easy to see that each of these integrals converges in a certain domain of values λ and is in this domain an analytic function of λ. So the integral $\mathscr{T}_\lambda$ is defined as an analytic function of λ. It can be shown that at $\lambda=0$ the function $\mathscr{T}_\lambda$ is regular and that $\mathscr{T}_0=K_\pi(g_1g_2\mid u,v)$.

4. Unitariness of the Operators $T_\pi(g)$. Let us show that *the operators $T_\pi(g)$ of the representations of the discrete series are unitary, that is,*

$$T_\pi^*(g)=T_\pi^{-1}(g)$$

where the asterisk denotes the adjoint operator.

For $T_\pi^*(g)$ is given by the kernel†

$$\overline{K_\pi(g\mid v,u)}$$

$$= a_\tau\bar{c}_\tau\,\frac{\operatorname{sign}_\tau\beta}{|\beta|}\operatorname{sign}_\tau v\cdot\chi\left(-\frac{\delta v+\alpha u}{\beta}\right)\int\chi\left(\frac{1}{\beta}\,(vt+ut^{-1})\right)\pi^{-1}(t)\,d^*t$$

$$= a_\tau c_\tau\frac{\operatorname{sign}_\tau(-\beta)}{|-\beta|}\operatorname{sign}_\tau u\cdot\chi\left(\frac{\alpha u+\delta v}{-\beta}\right)\int\chi\left(-\frac{1}{-\beta}(ut+vt^{-1})\right)\pi(t)\,d^*t^*).$$

† We use the relation $\bar{c}_\tau=c_\tau\operatorname{sign}_\tau(-1)$, see **§ 2.7**.

So we see that $\overline{K_\pi(g \mid v, u)} = K_\pi(g^{-1} \mid u, v)$, that is, $T_\pi^*(g) = T_\pi(g^{-1})$.

On the other hand, since the operators $T_\pi(g)$ form a representation, we have $T_\pi(g^{-1}) = T_\pi^{-1}(g)$. Therefore, $T_\pi^*(g) = T_\pi^{-1}(g)$, as required.

5. The π-Realization of the Representations of the Discrete Series. In this subsection and the next, we give two other realizations of the representations of the discrete series. We obtain the π-realization by going over from the functions $\varphi(u)$ to their Mellin transforms:

$$F(\pi_1) = \int \varphi(u)\pi_1^{-1}(u)\, |u|^{-\frac{1}{2}}\, du. \tag{1}$$

It is not hard to see that the kernels $\tilde{K}_\pi(g \mid \pi_1, \pi_2)$ of the operators $T_\pi(g)$ in the π-realization can be expressed in terms of their kernels $K_\pi(g \mid u, v)$ in the original realization of the representation by the following formula:

$$\tilde{K}_\pi(g \mid \pi_1, \pi_2) = \int K_\pi(g \mid u, v)\pi_1^{-1}(u)\, |u|^{-\frac{1}{2}}\pi_2(v)\, |v|^{-\frac{1}{2}}\, du\, dv. \tag{2}$$

Let us find the formulae in the π-realization for the representation operators corresponding to the matrices

$$\delta = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}, \qquad z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \qquad z \neq 0$$

and

$$\zeta = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}, \qquad \zeta \neq 0.$$

For this purpose we need, apart from the Gamma-function connected with $\mathbf{K}$, the Gamma-function $\Gamma_\tau(\pi)$ connected with $\mathbf{K}(\sqrt{\tau})$,

$$\Gamma_\tau(\pi) = \int_{\mathbf{K}^*(\sqrt{\tau})} \chi_\tau(t)\pi(t)\, d^*t.$$

Here π ranges over the set of multiplicative characters on $\mathbf{K}(\sqrt{\tau})$, and $\chi_\tau(t)$ is an additive character on $\mathbf{K}(\sqrt{\tau})$, which can be expressed in terms of the character $\chi(x)$ on $\mathbf{K}$ by the formula

$$\chi_\tau(t) = \chi(t + \bar{t}).$$

We assume that all multiplicative characters on $\mathbf{K}$ are extended to multiplicative characters on $\mathbf{K}(\sqrt{\tau})$; we denote the latter by the same letters as those used before. Furthermore, let $\bar{\pi}(t)$ denote the character corresponding to the formula

$$\bar{\pi}(t) = \pi(\bar{t}).$$

We show that *the representation operators corresponding to the matrices* δ, z, *and* ζ *are given in the* π*-realization by the following formulae*:

$$T_\pi(\delta)F(\pi_1) = \pi\pi_1^2\pi_\tau(\delta)F(\pi_1); \tag{3}$$

$$T_\pi(z)F(\pi_1) = \pi_1\pi_2^{-1}(z)\int \Gamma(\pi_1^{-1}\pi_2)F(\pi_2)\,d\pi_2; \tag{4}$$

$$T_\pi(\zeta)F(\pi_1) = \int \frac{\Gamma_\tau(\pi\pi_2\bar{\pi}_2\pi_0^2)}{\Gamma_\tau(\pi\pi_1\bar{\pi}_0\pi_0^2)}\,\Gamma(\pi_1\pi_2^{-1})\pi_1^{-1}\pi_2(-\zeta)F(\pi_2)\,d\pi_2, \tag{5}$$

where $\pi_0^2(t) = |t\bar{t}|^{\frac{1}{4}}$.

These formulae are similar to the formulae for the representations of the principal series obtained in **§ 3.2**. Indeed, according to **§ 3.2** the formula for the operator $T_\pi(z)$ of the representation of the principal series is precisely the same as (4), but the operator $T_\pi(\zeta)$ of the representation of the principal series is given by the following formula:

$$T_\pi(\zeta)F(\pi_1) = \int \frac{\Gamma(\pi_2\pi_0)\,\Gamma(\pi\pi_2\pi_0)}{\Gamma(\pi_1\pi_0)\,\Gamma(\pi\pi_1\pi_0)}\,\Gamma(\pi_1\pi_2^{-1})\pi_1^{-1}\pi_2(-\zeta)F(\pi_2)\,d\pi_2.$$

(3) and (4) are easily obtained on the basis of the formula (4) in **§ 4.1** for the kernel $K_\pi(g \mid u, v)$. We give a derivation for (5). The kernel of $T_\pi(\zeta)$ is given in the π-realization by the following formula:

$$\tilde{K}_\pi(\zeta \mid \pi_1, \pi_2) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \zeta}{|\zeta|}$$

$$\iint\limits_{t\bar{t}=v/u} \chi\left(\frac{1}{\zeta}\,(u+v-u(t+\bar{t}))\right)\operatorname{sign}_\tau u\cdot\pi_1^{-1}(u)|u|^{-\frac{1}{2}}\pi_2(v)\;|v|^{-\frac{1}{2}}\pi(t)\;d^*t\,du\,dv.$$

After elementary transformations we find

$$\tilde{K}_\pi(\zeta \mid \pi_1, \pi_2) = c_\tau\pi_1^{-1}\pi_2(\zeta)$$
$$\iint \chi(u(1-t)(1-\bar{t}))\operatorname{sign}_\tau u\pi_1^{-1}\pi_2(u)\pi\pi_2\bar{\pi}_2(t)\,du\,dt.$$

By integrating over u we obtain

$$\tilde{K}_\pi(\zeta \mid \pi_1, \pi_2) = c_\tau\pi_1^{-1}\pi_2(\zeta)\,\Gamma(\pi_1^{-1}\pi_2\pi_\tau\pi_0^2)$$
$$\int \pi\pi_2\bar{\pi}_2(t)\pi_1\bar{\pi}_1\pi_1^{-1}(1-t)\;|(1-t)(1-\bar{t})|^{-1}\,dt.$$

where

$$\pi_\tau(x) = \operatorname{sign}_\tau x, \qquad x \in \mathbf{K}^*,$$
$$\pi_0(t) = |t\bar{t}|^{\frac{1}{4}}, \qquad t \in \mathbf{K}^*(\sqrt{\tau}).$$

The last integral is the Beta-function connected with $\mathbf{K}(\sqrt{\tau})$ and can therefore be expressed in terms of Γ_τ. As a result we find

$$\tilde{K}_\pi(\zeta \mid \pi_1, \pi_2) = c_\tau \pi_1^{-1}\pi_2(\zeta) \cdot \Gamma(\pi_1^{-1}\pi_2\pi_\tau\pi_0^2) \frac{\Gamma_\tau(\pi\pi_2\bar{\pi}_2\pi_0^2)\,\Gamma_\tau(\pi_1\bar{\pi}_1\pi_2^{-1}\bar{\pi}_2^{-1})}{\Gamma_\tau(\pi\pi_1\bar{\pi}_1\pi_0^2)}.$$

Since according to **§ 2.10**

$$\Gamma_\tau(\pi_1\bar{\pi}_1\pi_2^{-1}\bar{\pi}_2^{-1}) = |\tau|^{-1}c_\tau\Gamma(\pi_1\pi_2^{-1}\pi_\tau)\,\Gamma(\pi_1\pi_2^{-1})$$
$$= |\tau|^{-1}c_\tau\pi_1\pi_2^{-1}\pi_\tau(-1)\frac{\Gamma(\pi_1\pi_2^{-1})}{\Gamma(\pi_1^{-1}\pi_2\pi_\tau\pi_0^2)},$$

we finally obtain (because $|\tau|^{-1}c_\tau^2\pi_\tau(-1) = 1$):

$$\tilde{K}_\pi(\zeta \mid \pi_1, \pi_2) = \pi_1^{-1}\pi_2(-\zeta)\frac{\Gamma_\tau(\pi\pi_2\bar{\pi}_2\pi_0^2)}{\Gamma_\tau(\pi\pi_1\bar{\pi}_1\pi_0^2)}\,\Gamma(\pi_1\pi_2^{-1}).$$

So the formula (5) for $T_\pi(\zeta)$ is proved.

Now we derive the formula for the operator $T_\pi(s)$, where $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We have

$$\tilde{K}_\pi(s \mid \pi_1, \pi_2) = a_\tau c_\tau \pi_\tau(-1)\int\int_{t\bar{t}=v/u} \chi(u(t+\bar{t}))\,\pi(t)\,\pi_1^{-1}\pi_\tau(u)\,\pi_2(v)\,|u|^{-1/2}\,|v|^{-1/2}\,d^*t\,du\,dv$$

$$= c_\tau\pi_\tau(-1)\int\int_{K(\sqrt{\tau})} \chi(u(t+\bar{t}))\,\pi_1^{-1}\pi_2\pi_\tau(u)\,\pi\pi_2\bar{\pi}_2\pi_0^{-2}(t)\,dt\,du.$$

By the change of variable $t = u^{-1}t'$ this integral reduces to the form

$$\tilde{K}_\pi(s \mid \pi_1, \pi_2) = c_\tau\pi_\tau(-1)\int\chi(t+\bar{t})\,\pi\pi_2\bar{\pi}_2\pi_0^{-2}(t)\,dt \cdot \int \pi_1^{-1}\pi_2^{-1}\pi_\tau\pi^{-1}\pi_0^{-2}(u)\,du,$$

that is,

$$\tilde{K}_\pi(s \mid \pi_1, \pi_2) = c_\tau\pi_\tau(-1)\,\Gamma_\tau(\pi\pi_2\bar{\pi}_2\pi_2^0)\,\delta(\pi_1^{-1}\pi_2^{-1}\pi_\tau\pi^{-1}), \tag{6}$$

where $\delta(\pi)$ is the Delta-function on the group of multiplicative characters of $\mathbf{K}$.

On the basis of (6) we obtain the following expression for $T_\pi(s)$:

$$T_\pi(s)F(\pi_1) = c_\tau\pi_\tau(-1)\,\Gamma_\tau(\bar{\pi}^{-1}\pi_1^{-1}\bar{\pi}_1^{-1}\pi_0^2)F(\hat{\pi}^{-1}\pi_\tau\pi_1^{-1}), \tag{7}$$

where π is the restriction of $\pi(t)$ to $\mathbf{K}$.

Incidentally, the operators $T_\pi(s)$ of the principal series are given in the π-realization by the similar formula

$$T_\pi(s)F(\pi_1) = \Gamma(\pi_1^{-1})\,\Gamma(\pi^{-1}\pi_1^{-1}\pi_0^2)F(\pi^{-1}\pi_1^{-1}).$$

6. Another Realization of the Representations of the Discrete Series. We now examine another realization of the representations of the discrete series, which we obtain by going over from the functions $\varphi(u)$ to their Fourier transforms

$$\tilde{\varphi}(x) = \int\varphi(u)\,\chi(ux)\,du.$$

In this realization the representation operator $T_\pi(g)$ is given by the kernel

$$K'_\pi(g \mid x, y) = \int K_\pi(g \mid u, v)\,\chi(ux - vy)\, du\, dv, \tag{1}$$

where $K_\pi(g \mid u, v)$ is the kernel of $T_\pi(g)$ in the original representation. We shall find an explicit expression for this kernel.

Let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\beta \neq 0$. When we substitute in (1) the expression (2) in **§ 4.1** for the kernel $K_\pi(g \mid u, v)$, we obtain

$$K'_\pi(g \mid u, v) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int \operatorname{sign}_\tau u \chi\left(\frac{\delta u + \alpha v}{\beta} - \frac{1}{\beta}(ut + vt^{-1})\right) \\ \times \pi(t)\,\chi(ux - vy)\, d^*t\, du\, dv. \tag{2}$$

Here the integration with respect to t is taken over the circle $t\bar{t} = vu^{-1}$. Substituting under the integral $v = ut\bar{t}$ we may rewrite this formula as follows:

$$K'_\pi(g \mid u, v) = c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int |u| \operatorname{sign}_\tau u \\ \times \chi\left[u\left(\frac{\delta + \alpha t\bar{t}}{\beta} - \frac{1}{\beta}(t + \bar{t}) + x - t\bar{t}y\right)\right]\pi(t)\, dt\, du, \tag{3}$$

where the integration with respect to t is taken over the whole plane $\mathbf{K}(\sqrt{\tau})$. Now we integrate with respect to u.

On the basis of the formula

$$\int \pi(u)\, |u|^{-1}\, \chi(ux)\, du = \Gamma(\pi)\pi^{-1}(x)$$

we obtain

$$K'_\pi(g \mid x, y) \\ = c_1 \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int \frac{\operatorname{sign}_\tau \left(\dfrac{\delta + \alpha t\bar{t}}{\beta} - \dfrac{1}{\beta}(t + \bar{t}) + x - t\bar{t}y\right)}{\left|\dfrac{\delta + \alpha t\bar{t}}{\beta} - \dfrac{1}{\beta}(t + \bar{t}) + x - t\bar{t}y\right|^2}\, \pi(t)\, dt, \tag{4}$$

where

$$c_1 = c_\tau \int |u| \operatorname{sign}_\tau u x(u)\, du.$$

Thus, *in the new realization the representations of the discrete series are constructed in the space of functions $\varphi(x)$ on K for which*

$$(\varphi, \varphi) = \int |\varphi(x)|^2\, dx < \infty.$$

The representation operator $T_\pi(g)$ *corresponding to the matrix* $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, *with* $\beta \neq 0$, *has the form*

$$T_\pi(g)\,\varphi(x) = \int K'_\pi(g \mid x, y)\,\varphi(y)\,dy, \tag{5}$$

where the kernel $K\ (g \mid x, y)$ *is given by* (4).

The formula for the kernels of the operators $T_\pi(g)$ corresponding to the triangular matrices $g = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ may be obtained from (4) by a limit process. However, it is more convenient to obtain it directly from the formulae for $T_\pi(g)$ in the original realization:

$$T_\pi(g)\,\varphi(u) = \operatorname{sign}_\tau \delta\pi(\delta)\ |\delta|\ \chi(\delta\gamma u)\,\varphi(\delta^2 u).$$

By applying the Fourier transform we easily find: the operator $T_\pi(g)$ corresponding to the matrix $g = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ has the following form in the new realization:

$$T_\pi(g)\,\varphi(x) = \operatorname{sign}_\tau \delta\pi(\delta)\ |\delta|\ \varphi\left(\frac{\alpha x + \gamma}{\delta}\right).$$

7. Equivalence of Representations of the Discrete Series. Each representation of the discrete series is given by a multiplicative character $\pi(t)$ on the plane $\mathbf{K}(\sqrt{\tau})$, and also by $\operatorname{sign}_\tau u$ (because it is realized either in the space of functions $\varphi(u)$ that are zero for $\operatorname{sign}_\tau u = -1$ or in the complementary space). We shall now find out which representations of the discrete series are equivalent.

1. If $\pi_1(t) \equiv \pi_2(t)$ on the circle $t\bar{t} = 1$, then the representations $T^+_{\pi_1}(g)$ and $T^+_{\pi_2}(g)$ (or $T^-_{\pi_1}(g)$ and $T^-_{\pi_2}(g)$, respectively) are equivalent.†

2. If $\pi_1(t) = \pi_2^{-1}(t)$, then the representations $T^+_{\pi_1}(g)$ and $T^+_{\pi_2}(g)$, or $T^-_{\pi_1}(g)$ and $T^-_{\pi_2}(g)$ respectively, are equivalent.

The converse statement follows from results in **§ 5.4**: if $\pi_1(t) \neq \pi_2(t)$ and $\pi_1(t) \neq \pi_2^{-1}(t)$ on the circle $t\bar{t} = 1$, then the representations $T^+_{\pi_1}(g)$ and $T^+_{\pi_2}(g)$ (or $T^-_{\pi_1}(g)$ and $T^-_{\pi_2}(g)$, respectively) are inequivalent.

3. The representations $T^+_{\pi_1}(g)$ and $T^-_{\pi_2}(g)$ are not equivalent for any π_1 and π_2.

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† We recall that we denote by $T^+_\pi(g)$ the representations realized in the subspace of functions $\varphi(u)$ that are zero for sign $u = -1$, and by $T^-_\pi(g)$ the representations realized in the complementary subspace.

Proof of Proposition 1. The kernel of the operator $T_\pi^+(g)$ has the form

$$K_\pi^+(g \mid u, v) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \int_{t\bar{t}=u/v} \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\, d^*t. \quad (1)$$

Here $\operatorname{sign}_\tau u = \operatorname{sign}_\tau v = 1$. Hence each of the elements u and v is either a square of an element from $\mathbf{K}$ or is of the form $\nu\bar{\nu}s^2$, where s is an element from $\mathbf{K}$ and ν a fixed element from $\mathbf{K}(\sqrt{\tau})$ such that $\nu\bar{\nu}$ is not a square of an element from $\mathbf{K}$.

We transform the formula for $K_\pi^+(g \mid u, v)$ and treat the cases $\pi(-1) = 1$ and $\pi(-1) = -1$ separately. Thus, let $\pi(-1) = 1$. If $u = s_1^2$, $v = s_2^2$, $s_1, s_2 \in \mathbf{K}$, then by the change of variable $t = \sqrt{\frac{v}{u}}\, t'$. we find

$$K_\pi^*(g \mid u, v) = \frac{\pi(\sqrt{v})}{\pi(\sqrt{u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int \chi\left(-\frac{\sqrt{uv}}{\beta}(t + \bar{t})\right)\pi(t)\, d^*t. \quad (2)$$

Since $\pi(x) = \pi(-x)$, all the factors in this expression are single-valued functions of u and v.

Similarly, if $u = \nu\bar{\nu}s_1^2$, $v = s_2^2$, $s_1, s_2 \in \mathbf{K}$, then $\left(\text{by the change of variable } t = \frac{\sqrt{v}}{\nu\sqrt{(\nu\bar{\nu})^{-1}u}}\, t'\right)$

$$K_\pi^+(g \mid u, v) = \frac{\pi(\sqrt{v})}{\pi(\nu\sqrt{(\nu\bar{\nu})^{-1}u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int_{t\bar{t}=1} \chi\left(-\frac{\sqrt{(\nu\bar{\nu})uv}}{\beta}\left(\frac{t}{\nu} + \frac{\bar{t}}{\bar{\nu}}\right)\right)\pi(t)\, d^*t; \quad (3)$$

if $u = s_1^2$, $v = \nu\bar{\nu}s_2^2$ $(s_1, s_2 \in \mathbf{K})$, then

$$K_\pi^+(g \mid u, v) = \frac{\pi(\nu\sqrt{(\nu\bar{\nu})^{-1}v})}{\pi(\sqrt{u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int_{t\bar{t}=1} \chi\left(-\frac{\sqrt{(\nu\bar{\nu})^{-1}uv}}{\beta}(\nu t + \bar{\nu}\bar{t})\right)\pi(t)\, d^*t. \quad (4)$$

Finally, if $u = \nu\bar{\nu}s_1^2$, $v = \nu\bar{\nu}s_2^2$, then

$$K_\pi^+(g \mid u, v) = \frac{\pi(\nu\sqrt{(\nu\bar{\nu})^{-1}v})}{\pi(\nu\sqrt{(\nu\bar{\nu})^{-1}u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int_{t\bar{t}=1} \chi\left(-\frac{\sqrt{uv}}{\beta}(t + \bar{t})\right)\pi(t)\, d^*t. \quad (5)$$

In the representation space we consider the operator A_π of multiplication by the function $a(u)$

$$A_\pi \varphi(u) = a(u)\,\varphi(u) \quad (6)$$

where $a(u) = \pi(\sqrt{u})$ when $u = s^2$, $s \in \mathbf{K}$, and $a(u) = \pi(\nu\sqrt{(\nu\bar{\nu})^{-1}u})$ when $u = \nu\bar{\nu}s^2$, $s \in \mathbf{K}$.

We transform $T_\pi^+(g)$ to the equivalent representation

$$\hat{T}_\pi^+(g) = A_\pi^{-1} T_\pi^+(g) A_\pi.$$

Clearly, the formulae for the kernels of the operators $T_\pi^+(g)$ are obtained from (2)–(5) by omitting the first factors. Hence, these kernels depend only on the values assumed by the character $\pi(t)$ on the circle $t\bar{t} = 1$. So we have shown that if $\pi_1(t) = \pi_2(t)$ on $t\bar{t} = 1$ and $\pi_1(-1) = 1$, then the representations $T_{\pi_1}^+(g)$ and $T_{\pi_2}^+(g)$ are equivalent.

Now we take the case $\pi(-1) = -1$. Let $\pi_0(t)$ be a fixed character such that $\pi_0(-1) = -1$. Just as in the first case, we then transform the formula for the kernel of $T_\pi^+(g)$ to the following form.

If $u = s_1^2$, $v = s_2^2$, $s_1, s_2 \in \mathbf{K}$, then

$$K_\pi^+(g \mid u, v) = \frac{\pi\pi_0^{-1}(\sqrt{v})}{\pi\pi_0^{-1}(\sqrt{u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \pi_0\left(\frac{\sqrt{v}}{\sqrt{u}}\right) \int_{t\bar{t}=1} \chi\left(-\frac{u}{\beta}\frac{\sqrt{v}}{\sqrt{u}}(t + \bar{t})\right)\pi(t)\, d^*t.$$

(The expression $\pi_0\left(\frac{\sqrt{v}}{\sqrt{u}}\right) \int_{t\bar{t}=1} \chi\left(-\frac{u}{\beta}\frac{\sqrt{v}}{\sqrt{u}}(t + \bar{t})\right)\pi(t)\, d^*t$ is a single-valued function of u and v, because it does not depend on the choice of the sign of $\sqrt{u}$ and $\sqrt{v}$.)

If $u = \nu\bar{\nu}s_1^2$, $v = s_2^2$, $s_1, s_2 \in \mathbf{K}$, then

$$K_\pi^+(g \mid u, v) = \frac{\pi\pi_0^{-1}(\sqrt{v})}{\pi\pi_0^{-1}(\nu\sqrt{(\nu\bar{\nu})^{-1}u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int_{t\bar{t}=1} \chi\left(-\frac{u}{\beta}\frac{\sqrt{v}}{\sqrt{(\nu\bar{\nu})^{-1}u}}\left(\frac{t}{\nu} + \frac{\bar{t}}{\bar{\nu}}\right)\right)\pi(t)\, d^*t$$

and so forth.

We go over from $T_\pi^+(g)$ to the equivalent representation $\hat{T}_\pi^+(g) = A_{\pi\pi_0^{-1}}^{-1} T_\pi^+(g) A_{\pi\pi_0^{-1}}$, where the operator A_π is given by (6).

Again, the kernel of the operators $\hat{T}_\pi^+(g)$ depends only on the values assumed by $\pi(t)$ on $t\bar{t} = 1$. Hence, if $\pi_1(t) = \pi_2(t)$ on $t\bar{t} = 1$, the representations $T_{\pi_1}^+(g)$ and $T_{\pi_2}^+(g)$ are equivalent. Proposition 1 is now proved.

Proof of Proposition 2. In the formula for the kernel of the operator $T_\pi^+(g)$

$$K_\pi^+(g \mid u, v) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \int_{t\bar{t}=v/u} \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right) \pi(t)\, d^*t$$

we make the change of variable $t = vu^{-1}t'^{-1}$, and obtain

$$K_\pi^+(g \mid u, v) = \frac{\pi(v)}{\pi(u)} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int_{t\bar{t}=v/u} \chi\left(-\frac{1}{\beta}(vt^{-1} + ut)\right) \pi^{-1}(t)\, d^*t,$$

that is,

$$K_\pi^+(g \mid u, v) = \frac{\pi(v)}{\pi(u)} K_{\pi^{-1}}^+(g \mid u, v). \tag{7}$$

The equivalence of $T_\pi^+(g)$ and $T_{\pi^{-1}}^+(g)$ follows immediately from (7).

Proof of Proposition 3. Let A be a bounded operator map the representation space of $T_{\pi_1}^+(g)$ into that of $T_{\pi_2}^-(g)$ and commuting with the representations:

$$T_{\pi_2}^-(g) A = A T_{\pi_1}^+(g). \tag{8}$$

Our task is to show that $A = 0$. We examine the operators $T_{\pi_1}^+(g)$ and $T_{\pi_2}^-(g)$ corresponding to the matrices $g = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$. These operators have the form

$$T_{\pi_1}^+(g)\varphi(u) = \chi(\gamma u)\varphi(u), \qquad T_{\pi_2}^-(g)\psi(u) = \chi(\gamma u)\varphi(u)$$

We set $\psi(u) = A\varphi(u)$. Then the condition (8) can be written in the form

$$\chi(\gamma u)\psi(u) = A[\chi(\gamma u)\varphi(u)]$$

for every γ in $\mathbf{K}$. Hence, it follows immediately that

$$f(u)\psi(u) = A[f(u)\varphi(u)] \tag{9}$$

for every bounded function $f(u)$ on $\mathbf{K}$. In particular, we consider the function

$$f(u) = \begin{cases} 1, & \text{when } \operatorname{sign}_\tau u = 1, \\ 0, & \text{when } \operatorname{sign}_\tau u = -1. \end{cases}$$

Since the functions $\varphi(u)$ are concentrated in the domain $\operatorname{sign}_\tau u = 1$ and $\psi(u)$ in the domain $\operatorname{sign}_\tau u = -1$, we have: $f(u)\varphi(u) = \varphi(u)$, $f(u)\psi(u) = 0$. Consequently, equation (9) gives us $A\varphi(u) = 0$, that is, $A = 0$.

All the representations of the discrete series $T_\pi^+(g)$ and $T_\pi^-(g)$ are irreducible.

The proof of this proposition follows the same line as that in the case of representations of the continuous series (see **§ 3.4**).

8. Discrete Series for the Field of 2-adic Numbers. In the preceding account we have assumed everywhere that the characteristic of the residue class field O/P is different from 2. However, in Chapter III we need the representations of the group of unimodular matrices of order 2 with elements from the field Q_2 of 2-adic numbers.

This case differs only insignificantly from the general case treated above. In fact, the constructions of the principal series, the supplementary series, and the special representation carry over to Q_2 without change. Some modifications are required only in the description of the discrete series, which we now indicate.

In the case $K = Q_2$ the factor group $\mathbf{K}^*/(\mathbf{K}^*)^2$ is of order 8 and can be represented as a direct sum of three cyclic groups of order 2. As generators of these groups we can take the cosets of $\mathbf{K}^*/(\mathbf{K}^*)^2$ containing the numbers 2, 3, and 5.

For from the arguments in **§ 1.5** it follows that the subgroup $A_2 \subset \mathbf{K}^*$ consisting of the elements of the form $1 + 8x$, $|x| \leq 1$, is contained in $(O^*)^2$. Also a direct computation shows that when $|x| = 1$, then $x^2 \in A_2$. Our statement on the structure of $\mathbf{K}^*/(\mathbf{K}^*)^2$ follows from this.

Thus, the field $K = Q_2$ has seven distinct quadratic extensions $\mathbf{K}(\sqrt{\tau})$, $\tau = 2, 3, 5, 6, 10, 15, 30$. It can be verified that in each of these extensions the subgroup $\mathbf{K}_\tau^*$ consisting of the elements of the form $z\bar{z}$, $z \in \mathbf{K}(\sqrt{\tau})$, is of index 2 in $\mathbf{K}^*$. Therefore, we may define the functions $\operatorname{sign}_\tau x$ that assume the values ± 1 and give a complete set of characters on $\mathbf{K}^*/(\mathbf{K}^*)^2$. The construction of the discrete series as described in this section may now be extended to the field Q_2, and here we obtain not three but seven discrete series of representations.

§ 5. THE TRACES OF IRREDUCIBLE REPRESENTATIONS OF G

1. Statement of the Problem. Let $T_\pi(g)$ be a representation of G belonging to the continuous (principal or supplementary) or

discrete series. With every finite function† $f(g)$ on G we associate the operator

$$T_\pi(f) = \int f(g)\, T_\pi(g)\, dg. \tag{1}$$

Then the following proposition holds.

The operator $T_\pi(f)$ has a trace, which we denote by Tr $T_\pi(f)$, *and this trace is a continuous functional in the space of finite functions $f(g)$.* So the trace Tr $T_\pi(g)$ of $T_\pi(g)$ is defined as a generalized function on G:

$$(\mathrm{Tr}\ T_\pi(g), f(g)) = \mathrm{Tr}\ T_\pi(f).$$

For the classical groups over the field of complex numbers this result was first obtained by Gel'fand and Naimark. Later it was proved by Godement and Harish-Chandra for the irreducible unitary representations of every real semisimple Lie group.

In the Appendix to this Chapter we give a proof of this proposition for the group of matrices of order 2 with elements from a disconnected locally compact topological field.

Our object is to compute the traces Tr $T_\pi(g)$ of the operators of the irreducible representations.

In this section we compute the traces Tr $T_\pi(g)$ on the basis of a unified method for all fields **K**.

We use the formula

$$\mathrm{Tr}\ T_\pi(g) = \int K_\pi(g \mid u, u)\, du.$$

where $K_\pi(g \mid u, v)$ is the kernel of this operator.

2. The Traces of the Representations of the Continuous Series. The representation operator of the continuous series corresponding to the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is given by the following formula (see **§ 3.1**):

$$T_\pi(g) f(x) = f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) \pi(\beta x + \delta)\, |\beta x + \delta|^{-1}.$$

Thus, $T_\pi(g)$ may be regarded as an integral operator whose kernel is the generalized function

$$K_\pi(g \mid x, y) = \pi(\beta x + \delta)\, |\beta x + \delta|^{-1}\, \delta\left(\frac{\alpha x + \gamma}{\beta x + \delta} - y\right). \tag{1}$$

† In the case of a connected field **K** we always assume that the function $f(g)$ is infinitely differentiable; in the case of a disconnected field $f(g)$ is assumed to be piecewise constant.

We compute the trace of $T_\pi(g)$ by the formula

$$\begin{aligned}\operatorname{Tr} T_\pi(g) &= \int K_\pi(g \mid x, x)\, dx \\ &= \int \pi(\beta x + \delta)\, |\beta x + \delta|^{-1}\, \delta\left(\frac{\alpha x + \gamma}{\beta x + \delta} - x\right) dx. \qquad (2)\end{aligned}$$

We may assume that $\beta \neq 0$ (otherwise we pass from g to any matrix conjugate to it). We make the change of variables $\beta x + \delta = t$, and obtain

$$\begin{aligned}\operatorname{Tr} T_\pi(g) &= \int \delta(\alpha + \delta - t - t^{-1})\pi(t)\, |t|^{-1}\, dt \\ &= \int \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1})\pi(t)\, |t|^{-1}\, dt, \qquad (3)\end{aligned}$$

where λ_g and λ_g^{-1} are the eigenvalues of g. From (3) it is clear that $\operatorname{Tr} T_\pi(g)$ *is concentrated on the matrices g whose eigenvalues λ_g and λ_g^{-1} lie in* $\mathbf{K}$, because the expression $\lambda_g + \lambda_g^{-1} - t - t^{-1}$, which is the argument of the Delta-function, vanishes at zero only for $t = \lambda_g$ and $t = \lambda_g^{-1}$.

The integral (3) is easy to evaluate. For this purpose it is sufficient to use the following property of the Delta-function:†

$$\delta((t - a)(t - b)) = \frac{1}{|a - b|}\,(\delta(t - a) + \delta(t - b)) \qquad (4)$$

(provided $a \neq b$). Suppose that λ_g and λ_g^{-1} lie in $\mathbf{K}$ and that $\lambda_g \neq \lambda_g^{-1}$. Then we have

$$\begin{aligned}|t|^{-1}\, \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1}) &= \delta((t - \lambda_g)(t - \lambda_g^{-1})) \\ &= \frac{1}{|\lambda_g - \lambda_g^{-1}|}\,(\delta(t - \lambda_g) + \delta(t - \lambda_g^{-1})).\end{aligned}$$

Substituting this expression in (3) we find

$$\operatorname{Tr} T_\pi(g) = \frac{\pi(\lambda_g) + \pi(\lambda_g^{-1})}{|\lambda_g - \lambda_g^{-1}|}. \qquad (5)$$

Thus, *the trace of the operator $T_\pi(g)$ of a representation of the continuous series is expressed by* (5), *provided the eigenvalues λ_g and λ_g^{-1} of g lie in* $\mathbf{K}$;

$$\operatorname{Tr} T_\pi(g) = 0,$$

when λ_g and λ_g^{-1} do not lie in $\mathbf{K}$.

From (5) it follows that the traces $\operatorname{Tr} T_{\pi_1}(g)$ and $\operatorname{Tr} T_{\pi_2}(g)$ of two representations of the continuous series coincide if and only if

† A proof of (4) for the field of real numbers is given in volume 7. We recommend that the reader prove (4) as an easy exercise in analysis in disconnected fields.

either $\pi_1 = \pi_2$ or $\pi_1 = \pi_2^{-1}$. Hence, we conclude: if $\pi_1 \neq \pi_2$ and $\pi_1 \neq \pi_2^{-1}$, *then the representations* $T_{\pi_1}(g)$ *and* $T_{\pi_2}(g)$ *of the continuous series are inequivalent.*

3. The Trace of the Singular Representation. The arguments in **§ 5.2** and the formula for the trace remain valid for the representations of the supplementary series, and also for the nonunitary representations in the spaces $\mathscr{D}_\pi$ (see **§ 3.8**).

We make use of this fact to compute the trace of the singular representation $T_0(g)$ in the case of a disconnected field.

We recall how the singular representation is constructed. We consider the space $\mathscr{D}_\pi$, $\pi(x) = |x|^{-1}$, of functions $f(x_1, x_2)$ satisfying the following condition of homogeneity:

$$f(tx_1, tx_2) = |t|^{-2} f(x_1, x_2) \tag{1}$$

for every $t \neq 0$. The representation operator $T_\pi(g)$ in $\mathscr{D}_\pi$ is given by the formula

$$T_\pi(g) f(x_1, x_2) = f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2). \tag{2}$$

If transfer from the homogeneous functions of two variables $f(x_1, x_2)$ to the functions of a single variable $\varphi(x) = f(x, 1)$, we obtain another realization of the space $\mathscr{D}_\pi$. In this realization the representation operator has the form

$$T_\pi(g)\, \varphi(x) = \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) |\beta x + \delta|^{-2}. \tag{3}$$

The space $\mathscr{D}_\pi$ contains an invariant subspace $\mathscr{F}_\pi$ consisting of the functions $\varphi(x)$ for which

$$\int \varphi(x)\, dx = 0.$$

The singular representation of G is a representation in the subspace† $\mathscr{F}_\pi$.

Clearly, the factor space $\mathscr{D}_\pi / \mathscr{F}_\pi$ is one-dimensional and the unit representation of the group acts in it. So the matrix of $T_\pi(g)$ in $\mathscr{D}_\pi$ has the form

$$\begin{pmatrix} 1 & * \\ 0 & T_0(g) \end{pmatrix},$$

where $T_0(g)$ is the operator of the singular representation.

Hence, it follows that we obtain the trace of $T_0(g)$ by computing the trace of the unit representation $\operatorname{Tr} T(g) \equiv 1$ from the trace of

† More accurately, not in the space $\mathscr{F}_\pi$ itself, but in its completion relative to the invariant scalar product.

the operator $T_\pi(g)$, $\pi(x) = |x|^{-1}$ defined by **§ 5.2** (5). As a result we find that: *the trace of the operator $T_0(g)$ of the singular representation is expressed by the following formula:*

$$\operatorname{Tr} T_0(g) = \frac{|\lambda_g| + |\lambda_g^{-1}|}{|\lambda_g - \lambda_g^{-1}|} - 1, \tag{4}$$

if the eigenvalues λ_g and λ_g^{-1} of g belong to $\mathbf{K}$;

$$\operatorname{Tr} T_0(g) = -1,$$

when λ_g and δ_g^{-1} do not belong to $\mathbf{K}$.

4. The Traces of the Representations of the Discrete Series. We recall that the operators $T_\pi^+(g)$ and $T_\pi^-(g)$ of the representations of the discrete series are given by the following formulae:

$$T_\pi^+(g)\,\varphi(u) = \int K_\pi(g \mid u, v)\,\varphi(v)\,dv, \quad \operatorname{sign}_\tau u = \operatorname{sign}_\tau v = 1,$$

$$T_\pi^-(g)\,\varphi(u) = \int K_\pi(g \mid u, v)\,\varphi(v)\,dv, \quad \operatorname{sign}_\tau u = \operatorname{sign}_\tau v = -1,$$

where,

$$K_\pi(g \mid u, v) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \operatorname{sign}_\tau u \cdot \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \int\limits_{tt = vu^{-1}} \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\,d^*t. \tag{1}$$

The representation $T_\pi^+(g)$ is realized in the space of functions on the half-line $\operatorname{sign}_\tau u = 1$, and $T_\pi^-(g)$ in the space of functions on the half-line $\operatorname{sign}_\tau u = -1$.

We compute the traces of the representations by the formulae

$$\begin{aligned}\operatorname{Tr} T_\pi^+(g) &= \int\limits_{\operatorname{sign}_\tau u = 1} K_\pi(g \mid u, u)\,du \\ &= a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int\limits_{\operatorname{sign}_\tau u = 1} \int\limits_{tt = 1} \chi\left(\frac{u}{\beta}(\alpha + \delta - t - t^{-1})\right)\pi(t)\,d^*t\,du,\end{aligned} \tag{2}$$

$$\begin{aligned}\operatorname{Tr} T_\pi^-(g) &= \int\limits_{\operatorname{sign}_\tau u = -1} K_\pi(g \mid u, u)\,du \\ &= -a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int\limits_{\operatorname{sign}_\tau u = -1} \int\limits_{tt = 1} \chi\left(\frac{u}{\beta}(\alpha + \delta - t - t^{-1})\right)\pi(t)\,d^*t\,du\end{aligned} \tag{2'}$$

It is convenient to compute not the trace of $T_\pi^+(g)$ and $T_\pi^-(g)$, but of their sum and difference.

First, we evaluate the difference of the traces. We have

$$\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g)$$
$$= a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int_K \int_{t\bar t=1} \chi\left(\frac{u}{\beta}(\alpha + \delta - t - t^{-1})\right)\pi(t)\, d^*t\, du.$$

Since

$$\int \chi(ux)\, du = \delta(x),$$

we obtain

$$\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g)$$
$$= a_\tau c_\tau \operatorname{sign}_\tau \beta \int_{t\bar t=1} \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1})\pi(t)\, d^*t, \quad (3)$$

where λ_g and λ_g^{-1} are the eigenvalues of g. From this formula it is clear that *the difference of the traces* $\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g)$ *is concentrated on those matrices g whose eigenvalues lie on the circle* $t\bar t = 1$ *on* $\mathbf{K}(\sqrt{\tau})$.

This holds because the argument of the Delta-function vanishes only for $t = \lambda_g$ and $t = \lambda_g^{-1}$.

Let us compute $\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g)$ for these matrices.

For this purpose we rewrite (3) as an integral over the whole plane $\mathbf{K}(\sqrt{\tau})$:

$$\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g)$$
$$= c_\tau \operatorname{sign}_\tau \beta \int \delta((t - \lambda_g) + (\bar t - \bar\lambda_g))\, \delta(1 - t\bar t)\pi(t)\, dt), \quad (4)$$

where the integral is taken over $\mathbf{K}(\sqrt{\tau})$.

We use the following relation:

$$\delta((t - \lambda_g) + (\bar t - \bar\lambda_g))\, \delta(1 - t\bar t)$$
$$= \frac{1}{|\tau|^{1/2}\, |\lambda_g - \lambda^{-1}|}(\delta_\tau(t - \lambda_g) + \delta_\tau(\bar t - \bar\lambda_g)), \quad (5)$$

where $\delta_\tau(t)$ is the Delta-function on $\mathbf{K}(\sqrt{\tau})$:

$$\delta_\tau(x + \sqrt{\tau}\, y) = \delta(x)\, \delta(y).$$

For if we set $t = x + \sqrt{\tau}\, y, \lambda_g = \alpha + \sqrt{\tau}\, \beta$, $\alpha^2 - \tau\beta^2 = 1$, we have

$$\delta((t - \lambda_g) + (\bar t - \bar\lambda_g))\delta(1 - t\bar t)$$
$$= \delta(x - \alpha)\delta(1 - x^2 + \tau y^2) = \delta(x - \alpha)\delta(\tau(y^2 - \beta^2))$$
$$= \frac{1}{|\tau|\, |\beta|}\delta(x - \alpha)(\delta(y - \beta) + \delta(y + \beta)).$$

Hence (5) follows immediately.

Substituting (5) in (4) we obtain

$$\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g) = c_\tau |\tau|^{-1/2} \operatorname{sign}_\tau \beta \frac{\pi(\lambda_g) + \pi(\lambda_g^{-1})}{|\lambda_g - \lambda_g^{-1}|}. \quad (6)$$

Thus, *the difference of the traces of the representations* $T_\pi^+(g)$ *and* $T_\pi^-(g)$ *of the discrete series corresponding to the quadratic extension* $\mathbf{K}(\sqrt{\tau})$ *of* $\mathbf{K}$ *is expressed by* (6), *if the eigenvalues* λ_g *and* λ_g^{-1} *of* g *lie on the circle* $t\bar{t} = 1$ *on* $K(\sqrt{\tau})$; *and*

$$\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g) = 0,$$

when λ_g *and* λ_g^{-1} *do not lie on this circle.*

Now we compute the trace of the sum $T_\pi(g) = T_\pi^+(g) \oplus T_\pi^{-1}(g)$ of the representations $T_\pi^+(g)$ and $T_\pi^-(g)$.

We have

$$\operatorname{Tr} T_\pi(g)$$
$$= a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int\limits_K \int\limits_{t\bar{t}=1} \operatorname{sign}_\tau u \, \chi\left(\frac{u}{\beta}(\lambda_g + \lambda_g^{-1} - t - t^{-1})\right) \pi(t) \, d^*t \, du.$$

Using the formula

$$\int \operatorname{sign}_\tau u \cdot \chi(ux) \, du = 2a_\tau^{-1} c_\tau^{-1} \frac{\operatorname{sign}_\tau x}{|x|}$$

(see **§ 2.7**), we find that *the trace of the sum* $T_\pi(g) = T_\pi^+(g) \oplus T_\pi^-(g)$ *of the representations of the discrete series is expressed by the following formula:*†

$$\operatorname{Tr} T_\pi(g) = \operatorname{Tr} T_\pi^+(g) + \operatorname{Tr} T_\pi^-(g)$$
$$= 2 \int\limits_{t\bar{t}=1} \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|} \pi(t) \, d^*t. \quad (7)$$

This formula is similar to the formula for the traces of the representations of the continuous series (see **§ 5.2**):

$$\operatorname{Tr} T_\pi(g) = \int\limits_K \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1}) \pi(t) \, d^*t$$

It is often useful to consider not the traces $\operatorname{Tr} T_\pi(g)$ themselves, but their Mellin transforms with respect to π, which we denote by $S(g; t)$. These Mellin

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† The integral (7) converges if λ_g and λ_g^{1-} do not lie on the circle $t\bar{t} = 1$. But if λ_g and λ_g^{-1} lie on this circle, then the integral must be understood in the sense of the regularizing value, namely as the value of the analytic function in ν:

$$f(\nu) = 2 \int\limits_{t\bar{t}=1} \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|^\nu} \pi(t) \, d^*t$$

at the point $\nu = 1$.

transforms have the following form. For representations of the continuous series

$$S(g; t) = \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1}), \qquad \text{where } t \in \mathbf{K}.$$

For representations of the discrete series corresponding to the quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$,

$$S(g; t) = 2\,\frac{\operatorname{sign}_\tau(\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|},$$

where t is a point of the circle $t\bar{t} = 1$ on the plane $\mathbf{K}(\sqrt{\tau})$.

Let us rewrite formula (7) for the field of real numbers in more detail. Here we have $t = e^{i\varphi}$, $d^*t = \frac{1}{2\pi}\,d\varphi$, $\pi(t) = e^{in\varphi}$ and (7) can easily be transformed as follows:

$$\operatorname{Tr} T_\pi(g) = \frac{1}{\pi i} \int\limits_C \frac{\zeta^n\,d\zeta}{(\zeta - \lambda_g)(\zeta - \lambda_g^{-1})}, \tag{8}$$

where the integration is taken over the unit circle C: $\zeta\bar{\zeta} = 1$. It is easy to evaluate this integral (for the final formula see **§ 5.5**). The integral (8) turns out to be different from zero both for complex and for real λ_g.

A different result holds for the case of a disconnected field $\mathbf{K}$. *Suppose that the eigenvalues λ_g and λ_g^{-1} of g do not lie on the circle $t\bar{t} = 1$ on $\mathbf{K}(\sqrt{\tau})$. Then*

$$\operatorname{Tr} T_\pi(g) = 0$$

for all π, except possibly a finite number of characters π (depending on g).

Proof. We expand $\lambda_g + \lambda_g^{-1}$ in a series (see **§ 1.3**)

$$x \equiv \lambda_g + \lambda_g^{-1} = \sum_{i=k}^{\infty} a_i \mathfrak{p}^i.$$

If $|x| > 1$, then

$$\operatorname{sign}_\tau(x - t - t^{-1}) = \operatorname{sign}_\tau x,\ |x - t - t^{-1}| = |x|$$

for every t on $t\bar{t} = 1$. Consequently,

$$\operatorname{Tr} T_\pi(g) = 2\,\frac{\operatorname{sign}_\tau x}{|x|} \int\limits_{t\bar{t}=1} \pi(t)\,d^*t = 0.$$

There remains the case† $|x| \leq 1$, that is, $k \geq 0$.

By hypothesis, for every t on $t\bar{t} = 1$ we have $t + t^{-1} \neq \lambda_g + \lambda_g^{-1}$. Therefore we can find a natural number m with the following property: if $t + t^{-1} = b_0 + b_1\mathfrak{p} + \cdots$, where t is an arbitrary point on $t\bar{t} = 1$, then $b_i \neq a_i$ for at least one index $i < m$.

We divide the circle $t\bar{t} = 1$ into a finite number of subsets

† We mention that if -1 is not a square in $\mathbf{K}$, then $|\lambda_g + \lambda_g^{-1}| \geq 1$.

$A_{b_0,\ldots,b_{m-1}}$; the subset $A_{b_0,\ldots,b_{m-1}}$ consists of all points t of the circle at which $t + t^{-1}$ has the first m given terms of the expansion: $b_0 + \cdots + b_{m-1}\mathfrak{p}^{m-1}$.

It is easy to see that $\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})$ and

$$|\lambda_g + \lambda_g^{-1} - t - t^{-1}|$$

are constant on each of these subsets. Therefore we consider the integrals

$$I_{b_0,\ldots,b_{m-1}} = \int\limits_{A_{b_0,\ldots,b_{m-1}}} \pi(t)\, d^*t.$$

and check that they are equal to zero for all π except a finite set.

On the circle $t\bar{t} = 1$ we consider the set A_m of points t of the form $t = 1 + \mathfrak{p}^m s$, where $|s| \leq 1$. It is not hard to see that A_m is a subgroup of finite index in the group of all points of the circle. Hence, there are only finitely characters on the circle that are identically equal to unity on A_m.

Suppose that the character π is not identically equal to unity on A_m. Then we show that for it $I_{b_0,\ldots,b_{m-1}} = 0$. For let $\pi(t_0) \neq 1$ for some $t_0 \in A_m$. Since the transformation $t \to tt_0$ preserves the set $A_{b_0,\ldots,b_{m-1}}$, we have

$$\pi(t_0) I_{b_0,\ldots,b_{m-1}} = \int\limits_{A_{b_0,\ldots,b_{m-1}}} \pi(tt_0)\, d^*t = \int\limits_{A_{b_0,\ldots,b_{m-1}}} \pi(t)\, d^*t = I_{b_0,\ldots,b_{m-1}}$$

Consequently, $I_{b_0,\ldots,b_{m-1}} = 0$, and the proposition is proved.

In this section we have computed the traces of the irreducible representations without detailed proofs. But there is no difficulty in giving a rigorous foundation to all the preceding calculations.

For example, let us look at the derivation of the formula for the trace of the sum $T_\pi(g) = T_\pi^+(g) \oplus T_\pi^-(g)$ of two representations of the discrete series. We assume $\mathbf{K}$ to be disconnected.

Let S be the space of finite piecewise constant functions on G. For every $f \in S$ the operator

$$T_\pi(f) = \int f(g) T_\pi(g)\, dg$$

is completely continuous (and positive if f is a function of the form $\varphi * \varphi^*$) and has a trace. We have to show that the trace of $T_\pi(f)$ is expressed by the formula

$$\operatorname{Tr} T_\pi(f) = 2 \int\limits_G \int\limits_{t\bar{t}=1} f(g) \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|} \pi(t)\, d^*t\, dg. \tag{9}$$

Since the kernel of $T_\pi(f)$ is

$$\int f(g) K_\pi(g \mid u, v)\, dg,$$

we have

$$\operatorname{Tr} T_\pi(f) = \lim_{k\to\infty} \int\limits_{|u|\leq q^k} \int\limits_G f(g) K_\pi(g \mid u, u)\, dg\, du$$

$$= \lim_{k\to\infty} \int\limits_G \int\limits_{|u|\leq q^k} f(g) K_\pi(g \mid u, u)\, du\, dg.$$

The interchange of the order of integration is permissible, because the integration with respect to G and to u is taken over a compact domain.

Substituting the explicit expression for $K_\pi(g \mid u, u)$ and integrating with respect to u we find

$$\operatorname{Tr} T_\pi(f)$$

$$= a_\tau c_\tau \lim_{k\to\infty} \int\limits_G f(g)\, \Gamma^{(k+s)}(\pi_\tau) \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\, \pi(t)\, d^*t\, dg. \tag{10}$$

where $\pi_\tau(x) = |x| \operatorname{sign}_\tau x$, and $\Gamma^{(n)}(\pi_\tau)$ is the incomplete Gamma-function:

$$\Gamma^{(n)}(\pi_\tau) = \int\limits_{|x|\leqslant q^n} \chi(x) \operatorname{sign}_\tau x\, dx.$$

The number s is defined by $|\lambda_g + \lambda_g^{-1} - t - t^{-1}| = q^s$.

It is easy to verify that the limit, as $k \to \infty$, of the sequence of generalized functions†

$$\varphi_k(g) = \int \Gamma^{(k+s)}(\pi_\tau) \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\, \pi(t)\, d^*t$$

is the generalized function

$$\Gamma(\pi_\tau) \int \frac{\operatorname{sign}_\tau(\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\, \pi(t)\, d^*t.$$

Thus, by passing to the limit in (10) and bearing in mind that $a_\tau c_\tau = 2\,\Gamma^{-1}(\pi_\tau)$ we obtain the required formula (9).

5. The Traces of the Representations of the Discrete Series for the Field of Real Numbers. For the field of real numbers a character on the unit circle is given by the formula

$$\pi(t) = e^{in\varphi}, \qquad t = e^{i\varphi}, \qquad 0 \leq \varphi \leq 2\pi. \tag{1}$$

Hence, a representation of the discrete series is given by an integer n, which we may assume to be positive (for negative n we obtain equivalent representations). We denote the representation operators by $T_n^+(g)$ and $T_n^-(g)$.

† The existence of the limit of the sequence $\varphi_n(g)$ follows from the existence of the trace $\operatorname{Tr} T_\pi(g)$, as a generalized function in S. Besides, it is not hard to prove the existence of this limit directly.

Formula (6) of **§ 5.4** gives us:

$$\operatorname{Tr} T_n^+(g) - \operatorname{Tr} T_n^-(g) = -i \operatorname{sign} \beta \frac{\lambda_g^n + \lambda_g^{-n}}{|\lambda_g - \lambda_g^{-1}|}. \tag{2}$$

when λ_g and λ_g^{-1} are complex numbers;

$$\operatorname{Tr} T_n^+(g) - \operatorname{Tr} T_n^-(g) = 0. \tag{3}$$

when λ_g and λ_g^{-1} are real numbers.

On the other hand, by formula (8) of **§ 5.4** we have:

$$\begin{aligned}\operatorname{Tr} T_n^+(g) + \operatorname{Tr} T_n^-(g) &= \frac{1}{\pi i} \int_C \frac{\zeta^n \, d\zeta}{(\zeta - \lambda_g)(\zeta - \lambda_g^{-1})} \\ &= \frac{2}{\lambda_g - \lambda_g^{-1}} \left(\frac{1}{2\pi i} \int_C \frac{\zeta^n \, d\zeta}{\zeta - \lambda_g^{-1}} - \frac{1}{2\pi i} \int_C \frac{\zeta^n \, d\zeta}{\zeta - \lambda_g} \right),\end{aligned} \tag{4}$$

where the integration is taken over the unit circle C: $\zeta\bar{\zeta} = 1$.

When λ_g and λ_g^{-1} are real numbers, one of them lies inside C, and the other outside. In this case we find by the Cauchy formula that

$$\operatorname{Tr} T_n^+(g) + \operatorname{Tr} T_n^-(g) = \frac{2\lambda_g^{-n}}{\lambda_g - \lambda_g^{-1}}, \tag{5}$$

where λ_g is the eigenvalue of g of greater absolute value.

When λ_g and λ_g^{-1} are complex numbers and, hence, lie on the unit circle, the integrals in (4) diverge and must be interpreted in the sense of regularlizing values.

We sketch this regularization without proof. We note that

$$\frac{1}{2\pi i} \int_C \frac{\zeta^n \, d\zeta}{\zeta - \lambda} = \begin{cases} \lambda^n, & \text{when } |\lambda| < 1, \\ 0, & \text{when } |\lambda| > 1. \end{cases}$$

Naturally, on the limit set $|\lambda| = 1$ the value of this integral must be defined by

$$\frac{1}{2\pi i} \int \frac{\zeta^n \, d\zeta}{\zeta - \lambda} = \frac{1}{2} \lambda^n.$$

So we obtain

$$\operatorname{Tr} T_n^+(g) + \operatorname{Tr} T_n^-(g) = -\frac{\lambda_g^n - \lambda_g^{-n}}{\lambda_g - \lambda_g^{-1}}, \tag{6}$$

when λ_g and λ_g^{-1} are complex numbers.

Now we have explicit formulae for $\operatorname{Tr} T_n^+(g) - \operatorname{Tr} T_n^-(g)$ and for $\operatorname{Tr} T_n^+(g) + \operatorname{Tr} T_n^-(g)$. We write down the formulae for $\operatorname{Tr} T_n^+(g)$ and $\operatorname{Tr} T_n^-(g)$, which follow immediately from them.

On the set of matrices g with real eigenvalues we have

$$\operatorname{Tr} T_n^+(g) = \operatorname{Tr} T_n^-(g) = \frac{\lambda_g^{-n}}{\lambda_g - \lambda_g^{-1}}, \tag{7}$$

where λ_g is the eigenvalue of greater absolute value.

On the set of matrices g with complex eigenvalues we have

$$\operatorname{Tr} T_n^+(g) = \frac{e^{-in\varphi}}{e^{i\varphi} - e^{-i\varphi}} \tag{8}$$

$$\operatorname{Tr} T_n^-(g) = \frac{e^{-in\varphi}}{e^{-i\varphi} - e^{i\varphi}} \tag{9}$$

where φ is determined from the condition that the matrix

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

is conjugate to g.

§ 6. THE INVERSION FORMULA AND THE PLANCHEREL FORMULA ON G

1. Statement of the Problem. Let $f(g)$ be a finite function† on G. With every unitary representation $T_\pi(g)$ of the continuous or discrete series of G we associate the operator

$$T_\pi(f) = \int f(g)\, T_\pi(g)\, dg. \tag{1}$$

The operator function $T_\pi(f)$, which is defined on the set of representations $T_\pi(g)$ of the continuous and the discrete series of a group, is called the Fourier transform of $f(g)$. Our task is to find an inversion formula for (1), that is, to express $f(g)$ in terms of its Fourier transform.

It is more convenient to state this problem in terms of generalized functions:‡ *to expand the Delta-function $\delta(g)$ on G by the traces of the representations of the continuous and the discrete series.* In other words, we have to find a function $\mu(\pi)$ on the set of representations such that

$$\delta(g) = \int \mu(\pi) \operatorname{Tr} T_\pi(g)\, d\pi. \tag{2}$$

The integral is taken over the set of representations of the continuous and the discrete series.

Note that the representations $T_\pi(g)$ and $T_{\pi^{-1}}(g)$ are equivalent, so that $\operatorname{Tr} T_\pi(g) = \operatorname{Tr} T_{\pi^{-1}}(g)$. By virtue of this fact, the function

† For a connected field we assume that $f(g)$ is infinitely differentiable; for a disconnected field, that $f(g)$ is constant in sufficiently small domains on G.

‡ The generalized function $\delta(g)$ is defined as follows: $(\delta(g), f(g)) = f(e)$, where e is the unit element of the group.

$\mu(\pi)$ in (2) is not uniquely determined. It is natural to impose on the required function $\mu(\pi)$ the additional condition:

$$\mu(\pi) = \mu(\pi^{-1}).$$

The inversion formula for the function $f(g)$ on G and the Plancherel formula, which we are looking for, are immediate consequences of (2). For, *let $f(g)$ be a finite function on the group belonging to the space of basic functions. Then* (2) *leads to the inversion formula*

$$f(g_0) = \int \mu(\pi) \operatorname{Tr} (T_\pi(f) T_\pi^{-1}(g_0))\, d\pi \tag{3}$$

and to the Plancherel formula

$$\int |f(g)|^2 = \int \mu(\pi) \operatorname{Tr} (T_\pi(f) T_\pi^*(f))\, d\pi, \tag{4}$$

where T^* denotes the adjoint operator.

For by multiplying both sides of (2) by $f(gg_0)$ and integrating over g we find

$$f(g_0) = \int \mu(\pi) \operatorname{Tr}\left(\int f(gg_0) T_\pi(g)\, dg\right) d\pi.$$

After the change of variable $gg_0 = g_1$ we arrive precisely at (3).

To obtain the Plancherel formula we apply (3) to the function

$$F(g) = \int f(g_1)\overline{f(g_1 g^{-1})}\, dg_1.$$

For $g = e$ we find

$$F(e) = \int \mu(\pi) \operatorname{Tr} T_\pi(F)\, d\pi. \tag{5}$$

It is easy to check that

$$T_\pi(F) = T_\pi(f) \cdot T_\pi^*(f) \tag{6}$$

On the other hand, we have

$$F(e) = \int |f(g)|^2\, dg. \tag{7}$$

Substituting (6) and (7) in (5) we obtain the required Plancherel formula.

Thus, our main task is to find the expansion of $\delta(g)$ by the traces of the irreducible representations

$$\delta(g) = \int \mu(\pi) \operatorname{Tr} T_\pi(g)\, d\pi. \tag{8}$$

This problem will be solved in **§ 6.2** for a disconnected field, and in **§ 6.5** for a connected field.

We give another expression for (8), by going over from the functions $\mu(\pi)$ and $\operatorname{Tr} T_\pi(g)$ to their Mellin transforms. For

representations $T_\pi(g)$ of the continuous series we set

$$S(g; t) = \int \operatorname{Tr} T_\pi(g)\pi(t)\, d\pi, \tag{9}$$

$$\varphi(t) = \int \mu(\pi)\pi(t)\, d\pi, \tag{10}$$

where $t \in \mathbf{K}$ and the integral is taken over the group of multiplicative characters on $\mathbf{K}$.

For representations $T_{\pi_\tau}(g)$ of the discrete series corresponding to the extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$ we set

$$S_\tau(g; t) = \int \operatorname{Tr} T_{\pi_\tau}(g)\pi(t)\, d\pi_\tau, \tag{9'}$$

$$\varphi_\tau(t) = \int \mu(\pi_\tau)\pi_\tau(t), \tag{10'}$$

where t is a point on the circle $t\bar{t} = 1$ in the plane $\mathbf{K}(\sqrt{\tau})$ and the integral is taken over the group of characters π_τ on $t\bar{t} = 1$.

Then (8) takes the form

$$\delta(g) = \int \varphi(t)S(g; t)\, d^*t + \sum_\tau \int_{t\bar{t}=1} \varphi_\tau(t)S_\tau(g; t)\, d^*t + a \operatorname{Tr} T_0(g). \tag{11}$$

Here the sum is over the set of discrete series of G (hence, for a connected field it consists of three terms: $\tau = \mathfrak{p}$, $\varepsilon\mathfrak{p}$, and ε).

The last term in (11) is the trace of the singular representation of G (see **§ 3.7**); it occurs only for a disconnected field $\mathbf{K}$.

The traces of the representations of the continuous and the discrete series, and also their Mellin transforms, were found in **§ 5.** Substituting the expressions for $S(g; t)$ and $S_\tau(g; t)$ (see **§ 5.4**) we obtain the inversion formula in the following form:

$$\delta(g) = \theta(g)\, \frac{\varphi(\lambda_g) + \varphi(\lambda_g^{-1})}{|\lambda_g - \lambda_g^{-1}|} + a\left[\theta(g)\, \frac{|\lambda_g| + |\lambda_g^{-1}|}{\lambda_g - \lambda_g^{-1}} - 1\right]$$
$$+ 2 \sum_\tau \int_{t\bar{t}=1} \varphi(t)\, \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\, d^*t, \tag{12}$$

where λ_g and λ_g^{-1} are the eigenvalues of g; $\theta(g) = 1$ when λ_g, $\lambda_g^{-1} \in \mathbf{K}$, and otherwise $\theta(g) = 0$.

The functions $\varphi(t)$ and $\varphi_\tau(t)$ and the coefficient a are so far not determined; we have to find them.

2. The Inversion Formula for a Disconnected Field. Suppose that the elements of the matrices of G belong to a disconnected field $\mathbf{K}$. We denote by $T_\pi(g)$ the representations of the continuous series of G, by $T_0(g)$ its special representation (**§ 3.1** and **§ 3.8**), and by

$T_{\pi_\tau}(g)$ the representations of the discrete series corresponding to the extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$ ($\tau = \mathfrak{p}$, $\varepsilon\mathfrak{p}$, or ε). Here we derive the following *inversion formula*:

$$c\delta(g) = \int \mu(\pi) \operatorname{Tr} T_\pi(g)\, d\pi + 2 \operatorname{Tr} T_0(g) + \sum_\tau \int \mu(\pi_\tau) \operatorname{Tr} T_{\pi_\tau}(g)\, d\pi_\tau, \quad (1)$$

where†

$$\mu(\pi) = -\int_K \pi(t)\, |1 - t|^{-2}\, dt, \quad (2)$$

$$\mu(\pi_\varepsilon) = -\int_{t\bar{t}=1} \pi(t)\, |1 - t|^{-2}\, d^*t, \quad (2')$$

$$\mu(\pi_\tau) = -\int_{t\bar{t}=1, |1-t|<1} \pi(t)[|1 - t|^{-2} + 1]\, d^*t, \quad \tau = \mathfrak{p},\ \varepsilon\mathfrak{p} \quad (2'')$$

$c = \dfrac{2(q + 1)}{q^2}$. (The value of the constant c will be computed in **§ 6.4**). The integrals (2′) and (2″) are taken over the circle $t\bar{t} = 1$ in $K(\sqrt{\varepsilon})$ and $K(\sqrt{\tau})$, $\tau = \mathfrak{p}$ or $\varepsilon\mathfrak{p}$, respectively.

Note that all the integrals (2)–(2″) diverge so that they have to be understood in the sense of the regularizing value. For example, $\mu(\pi)$ is the value of the analytic function in ν,

$$\varphi(\nu) = -\int \pi(t)\, |1 - t|^\nu\, dt,$$

for $\nu = -2$ (see **§ 2.6**).

First of all we substitute in (1) the expressions for the traces of the representations and pass to the Mellin transforms with respect to π (see **§ 6.1**). As a result the formula assumes the form

$$\begin{aligned} c\delta(g) = {} & -\theta(g) \frac{2\, |\lambda_g|}{|\lambda_g - \lambda_g^{-1}|\, |1 - \lambda_g|^2} \\ & + 2\left(\theta(g) \frac{|\lambda_g| + |\lambda_g^{-1}|}{|\lambda_g - \lambda_g^{-1}|} - 1\right) \\ & - 2 \int_{t\bar{t}=1} \frac{\operatorname{sign}_\varepsilon (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|\, |1 - t|^2}\, d^*t \\ & - 2 \sum_{\tau=\mathfrak{p},\varepsilon\mathfrak{p}} \int_{\substack{|1-t|<1 \\ t\bar{t}=1}} \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|\, |1 - t|^2}\, d^*t \\ & - 2 \sum_{\tau=\mathfrak{p},\varepsilon\mathfrak{p}} \int_{t\bar{t}=1, |1-t|<1} \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\, d^*t. \quad (3) \end{aligned}$$

† The norm $|t|$ in an expression of $\mathbf{K}$ is defined by $|t| = |t\bar{t}|^{1/2}$.

Here λ_g and λ_g^{-1} are the eigenvalues of g; $\theta(g) = 1$, when λ_g, $\lambda_g^{-1} \in \mathbf{K}$, $\theta(g) = 0$ otherwise.

The derivation of (3) is given in two stages. First, we check that the expression on the right-hand side of (3), which we denote briefly by $I(g)$, is zero for $\lambda_g = \pm 1$. Then we show that $I(g)$ is zero for all $g \neq e$ so that $I(g)$ is concentrated at the point $g = e$. It follows immediately that $I(g) = c\delta(g)$. The coefficient c will be computed in **§ 6.4.**

The fact that $I(g) = 0$ for $\lambda_g \neq \pm 1$ can be verified directly, by computing the integrals in (3). Here we have to discuss separately the following possible cases:

1. $\lambda_g \in \mathbf{K}$, $|\lambda_g| \neq 1$,
2. $\lambda_g \in \mathbf{K}$, $|\lambda_g| = 1$, $|\lambda_g - 1| = |\lambda_g + 1| = 1$,
3. $\lambda_g \in \mathbf{K}$, $|\lambda_g| = 1$, $|\lambda_g - 1| < 1$,
4. $\lambda_g \in \mathbf{K}$, $|\lambda_g| = 1$, $|\lambda_g + 1| < 1$,
5. $\lambda_g \in \mathbf{K}(\sqrt{\tau})$, $\tau = \mathfrak{p}, \varepsilon\mathfrak{p}$, $|\lambda_g - 1| < 1$,
6. $\lambda_g \in \mathbf{K}(\sqrt{\tau})$, $\tau = \mathfrak{p}, \varepsilon\mathfrak{p}$, $|\lambda_g + 1| < 1$,
7. $\lambda_g \in \mathbf{K}(\sqrt{\varepsilon})$, $|\lambda_g - 1| < 1$,
8. $\lambda_g \in \mathbf{K}(\sqrt{\varepsilon})$, $|\lambda_g + 1| < 1$,
9. $\lambda_g \in \mathbf{K}(\sqrt{\varepsilon})$, $|\lambda_g - 1| = |\lambda_g + 1| = 1$,

Below we give a table of the values of the integrals occurring in (3). The evaluation of some of these integrals will be given in **3.** The detailed verification of the fact that $I(g) = 0$ for $\lambda_g \neq \pm 1$ is left to the reader.

Derivation of the formulae. Notation: λ and λ^{-1} are the eigenvalues of g; $\nu = \lambda + \lambda^{-1} - 2$; q is the order of the finite residue class field O/P associated with $\mathbf{K}$ (see **§ 1.3**); $\left(\frac{a}{q}\right)$ is the Legendre symbol ($a \neq 0$ is an element of the finite field F of order q): $\left(\frac{a}{q}\right) = 1$ if a is the square of an element of F; $\left(\frac{a}{q}\right) = -1$ if a is not a square. $\left(\frac{-1}{q}\right) = 1$ when $q \equiv 1 \bmod 4$; $\left(\frac{-1}{q}\right) = -1$ when $q \equiv 3 \bmod 4$.

1. Value of the integral

$$I_\tau^{(1)}(\lambda) = \int\limits_{t\bar{t}=1, |1-t|<1} \frac{\operatorname{sign}_\tau(\lambda + \lambda^{-1} - t - t^{-1})}{|\lambda + \lambda^{-1} - t - t^{-1}|\,|1 - t|^2}\, d^*t$$

If $|\nu| \geq 1$, then

$$I_\tau^{(1)}(\lambda) = c'_\tau \frac{\operatorname{sign}_\tau \nu}{|\nu|}$$

where $c'_\tau = -\frac{1}{2}$ for $\tau = \mathfrak{p}$ or $\varepsilon\mathfrak{p}$; $c'_\varepsilon = -\dfrac{q}{q+1}$.

If $|\nu| < 1$, then the values $I_\tau^{(1)}(\lambda)$ are given below:

a. $\lambda \in \mathbf{K}$

$$I_{\mathrm{p}}^{(1)}(\lambda) = I_{\varepsilon\mathrm{p}}^{(1)}(\lambda) = -\frac{q^2+1}{2(q^2+q+1)}|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon}^{(1)}(\lambda) = -\frac{q}{q^2+q+1}|\nu|^{-3/2} - \frac{q^3}{(q+1)(q^2+q+1)}.$$

b. λ a point on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\mathrm{p}})$

$$I_{\mathrm{p}}^{(1)}(\lambda) = -\frac{1}{2\sqrt{q}}\left(\frac{q^2+q}{q^2+q+1} + \left(\frac{-1}{q}\right)(q+1)\right)|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon\mathrm{p}}^{(1)}(\lambda) = -\frac{1}{2\sqrt{q}}\left(\frac{q^2+q}{q^2+q+1} - \left(\frac{-1}{q}\right)(q+1)\right)|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon}^{(1)}(\lambda) = \frac{q^2+q}{\sqrt{q}\,(q^2+q+1)}|\nu|^{-3/2} - \frac{q^3}{(q+1)(q^2+q+1)}.$$

c. λ a point on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\varepsilon\mathrm{p}})$

$$I_{\mathrm{p}}^{(1)}(\lambda) = -\frac{1}{2\sqrt{q}}\left(\frac{q^2+q}{q^2+q+1} - \left(\frac{-1}{q}\right)(q+1)\right)|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon\mathrm{p}}^{(1)}(\lambda) = -\frac{1}{2\sqrt{q}}\left(\frac{q^2+q}{q^2+q+1} + \left(\frac{-1}{q}\right)(q+1)\right)|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon}^{(1)}(\lambda) = \frac{q^2+q}{\sqrt{q}\,(q^2+q+1)}|\nu|^{-3/2} - \frac{q^3}{(q+1)(q^2+q+1)}.$$

d. λ a point on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\varepsilon})$

$$I_{\mathrm{p}}^{(1)}(\lambda) = I_{\varepsilon\mathrm{p}}^{(1)}(\lambda) = \frac{(q+1)^2}{2(q^2+q+1)}|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon}^{(1)}(\lambda) = -\frac{(q+1)^2}{q^2+q+1}|\nu|^{-3/2} - \frac{q^3}{(q+1)(q^2+q+1)}.$$

2. Value of the integral

$$I_\tau^{(2)}(\lambda) = \int\limits_{t\bar{t}=1,\,|1-t|<1} \frac{\operatorname{sign}_\tau(\lambda+\lambda^{-1}-t-t^{-1})}{|\lambda+\lambda^{-1}-t-t^{-1}|}\,d^*t.$$

If $|\nu| \geq 1$, then

$$I_\tau^{(2)}(\lambda) = c''_\tau \frac{\operatorname{sign}_\tau \nu}{|\nu|},$$

where $c''_\tau = 1/2$ for $\tau = \mathrm{p}$ or $\varepsilon\mathrm{p}$; $c''_\varepsilon = \dfrac{1}{q+1}$.

If $|\nu| < 1$, then the value $I_\tau^{(2)}(\lambda)$ is given below:

a. $\lambda \in \mathbf{K}$

$$I_{\mathrm{p}}^{(2)}(\lambda) = I_{\varepsilon\mathrm{p}}^{(2)}(\lambda) = |\nu|^{-1/2} - 1/2, \qquad I_{\varepsilon}^{(2)}(\lambda) = |\nu|^{-1/2} - \frac{q}{q+1}.$$

b. λ a point on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\tau})$, $\tau = \mathfrak{p}$, $\varepsilon\mathfrak{p}$, or ε

$$I_{\mathfrak{p}}^{(2)}(\lambda) = I_{\varepsilon\mathfrak{p}}^{(2)}(\lambda) = -\tfrac{1}{2}, \qquad I_{\varepsilon}^{(2)}(\lambda) = -\frac{q}{q+1}.$$

3. Value of the integral

$$I_\varepsilon^\beta(\lambda) = \int\limits_{A_\beta} \frac{\operatorname{sign}_\varepsilon(\lambda + \lambda^{-1} - t - t^{-1})}{|\lambda + \lambda^{-1} - t - t^{-1}|}\, d^*t,$$

where the integral is taken over the component A_β of the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\varepsilon})$ defined by the condition $|t - \beta| < 1$. Here β is a point of the circle for which $|\beta + 1| = |\beta - 1| = 1$:

$$I_\varepsilon^\beta(\lambda) = \frac{1}{q-1}\frac{\operatorname{sign}_\varepsilon \nu}{|\nu|} \qquad \text{when} \quad |\lambda| \neq 1,$$

$I_\varepsilon^\beta(\lambda) = -\dfrac{q-1}{2(q+1)}$, when λ is a point on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\varepsilon})$ and either $|\lambda - \beta| < 1$ or $|\lambda - \bar{\beta}| < 1$, $I_\varepsilon^\beta(\lambda) = \dfrac{1}{q+1}$ in all the remaining cases.

Now we have to verify that $I(g) = 0$ for all the matrices $g \neq e$.

Observe that the integrals in (3) reduce to one of the following forms $a\,|\nu|^{-3/2} + b$, $a\,|\nu|^{-1/2} + b$, $a\,|\nu'|^{-1/2} + b$, where

$$\nu = \lambda_g + \lambda_g^{-1} - 2, \qquad \nu' = \lambda_g + \lambda_g^{-1} + 2$$

(see the derivation of the formulae). In fact, they can all be simplified. The cases $\lambda_g = 1$ and $\lambda_g = -1$ are special, because then $\nu = 0$ and $\nu' = 0$, respectively. Therefore they require a separate investigation. Let us show that the functions $|\nu|^{-3/2}$, $|\nu|^{-1/2}$ and $|\nu'|^{-1/2}$, regarded as generalized functions on the group, have no singularity for $g \neq e$. In other words, $(|\nu|^{-3/2}, f)$, $(|\nu|^{-1/2}, f)$ and $(|\nu'|^{-1/2}, f)$ are continuous functionals in the subspace of finite functions f on G that are equal to zero in a neighborhood of e.† Hence, it follows easily that the generalized function $I(g)$ is concentrated at e.

It is not hard to check that the integrals

$$(|\nu|^{-1/2}, f) = \int |\nu|^{-1/2} f(g)\, dg$$

and

$$(|\nu'|^{-1/2}, f) = \int |\nu'|^{-1/2} f(g)\, dg$$

converge in the usual sense for every finite function $f(g)$. Therefore it is sufficient to treat the integral

$$(|\nu|^{-3/2}, f) = \int |\nu|^{-3/2} f(g)\, dg.$$

† We recall that when we speak of finite functions, we assume in addition that the functions are "piecewise constant," that is, that they are constant in a sufficiently small neighborhood of every point g.

This integral must be understood in the sense of the regularizing value: $(|\nu|^{-3/2}, f)$ is the value of the analytic function of s

$$\varphi(s) = \int |\nu|^s f(g)\, dg \tag{4}$$

for $s = -3/2$. Our aim is to show that if $f(g) = 0$ in a neighborhood of $g = e$, then the function $\varphi(s)$ has no singularity for $s = -3/2$.

We may assume, without loss of generality, that $f(g)$ is concentrated in a sufficiently small neighborhood of a matrix $g_0 \neq e$ with the eigenvalues $\lambda_g = \lambda_g^{-1} = 1$.

We introduce a coordinate system in this neighborhood. The matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ belonging to this neighborhood has at least one of the elements β or γ different from zero, say $\gamma \neq 0$. Then we can take as coordinates in a neighborhood of g_0 the values γ, α, and $\nu = \alpha + \delta - 2$. In these coordinates the formula (4) for $\varphi(s)$ takes the form

$$\varphi(s) = \int |\nu|^s f(\nu, \alpha, \gamma) \frac{d\alpha\, d\gamma}{|\gamma|}\, d\nu.$$

But we know from **§ 1.3** that the only singularity of the generalized function $|\nu|^s$ is a pole at $s = -1$. Consequently, $\varphi(s)$ has no singularity at $s = -3/2$.

So we have shown that the generalized function $I(g)$ on the right side of (3) is concentrated at $g = e$. Hence it follows that $I(g) = c\delta(g)$, where c is a constant (see **§ 2.2**), and the inversion formula (1) is proved.

3. Computation of Certain Integrals. We show how to compute the integrals that occur in the derivation of the formulae in **§ 6.2**. We take as an example

$$I_{\mathrm{p}}^{(1)}(\lambda) = \int\limits_{t\bar{t}=1,\ |t-1|<1} \frac{\operatorname{sign}_{\mathrm{p}} (\lambda + \lambda^{-1} - t - t^{-1})}{|\lambda + \lambda^{-1} - t - t^{-1}|\, |1 - t|^2}\, d^*t \tag{1}$$

If $|\lambda + \lambda^{-1} - 2| > 1$, then for every t, $t\bar{t} = 1$, $|1 - t| < 1$, we have $\operatorname{sign}_\tau (\lambda + \lambda^{-1} - t - t^{-1}) = \operatorname{sign}_\tau (\lambda + \lambda^{-1} - 2)$, $|\lambda + \lambda^{-1} - t - t^{-1}| = |\lambda + \lambda^{-1} - 2|$, so that the integral (1) simplifies considerably:

$$I_{\mathrm{p}}^{(1)}(\lambda) = \frac{\operatorname{sign}_{\mathrm{p}} (\lambda + \lambda^{-1} - 2)}{|\lambda + \lambda^{-1} - 2|} \int \frac{d^*t}{|1 - t|^2}.$$

We give the details of the most complicated case when $|\lambda + \lambda^{-1} - 2| < 1$; then we have $|\lambda| = 1$ and $|\lambda - 1| < 1$. Suppose, for the sake of definiteness, that λ lies on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\mathrm{p}})$. In the other possible cases the integral is computed similarly. According to **§ 1.8** the elements of the circle $t\bar{t} = 1$,

$|1 - t| < 1$ in $\mathbf{K}(\sqrt{\mathrm{p}})$ have the following parametric representations: †

$$t = \frac{1 + \sqrt{\mathrm{p}}x}{1 - \sqrt{\mathrm{p}}x} = \frac{1 + \mathrm{p}x^2}{1 - \mathrm{p}x^2} + \sqrt{\mathrm{p}}\,\frac{2x}{1 - \mathrm{p}x^2},$$

where x ranges over all the integers in $\mathbf{K}$ (that is, $|x| \leq 1$). It is easy to check that $d^*t = \frac{1}{2}dx$, where dx is the invariant measure on $\mathbf{K}^+$.

We can represent λ in the same form:

$$\lambda = \frac{1 + \sqrt{\mathrm{p}}x_0}{1 - \sqrt{\mathrm{p}}x_0}$$

Substituting these expressions in (1) and transmitting the variable t to x, we find

$$I_{\mathrm{p}}^{(1)}(\lambda) = \frac{1}{2} \int\limits_{|x| \leq 1} \frac{\operatorname{sign}_{\mathrm{p}}\left(2\,\dfrac{1 + \mathrm{p}x_0^2}{1 - \mathrm{p}x_0^2} - 2\,\dfrac{1 + \mathrm{p}x^2}{1 - \mathrm{p}x^2}\right)}{\left|2\,\dfrac{1 + \mathrm{p}x_0^2}{1 - \mathrm{p}x_0^2} - 2\,\dfrac{1 + \mathrm{p}x^2}{1 - \mathrm{p}x^2}\right| \cdot \left|\dfrac{2\sqrt{\mathrm{p}}x}{1 - \sqrt{\mathrm{p}}x}\right|^2}\, dx.$$

This expression can be simplified considerably, because the functions $\operatorname{sign}_p x$ and $|x|$ depend only on the first terms of the expansion of x. We obtain

$$I_{\mathrm{p}}^{(1)}(\lambda) = \frac{1}{2} \int\limits_{|x| \leq 1} \frac{\operatorname{sign}_{\mathrm{p}}(\mathrm{p}x_0^2 - \mathrm{p}x^2)}{|\mathrm{p}x_0^2 - \mathrm{p}x^2|\,|\mathrm{p}x^2|}\, dx. \tag{2}$$

Now let us compute this integral. First we add to, and subtract from, $I_{\mathrm{p}}^{(1)}(\lambda)$ the integral

$$\frac{1}{2} \int\limits_{|x| > q^k} \frac{\operatorname{sign}_{\mathrm{p}}(\mathrm{p}x_0^2 - \mathrm{p}x^2)}{|\mathrm{p}x_0^2 - \mathrm{p}x^2|\,|\mathrm{p}x^2|}\, dx = \frac{1}{2} \int \frac{dx}{|\mathrm{p}x^2|^2}.$$

After elementary transformations we obtain

$$I_{\mathrm{p}}^{(1)}(\lambda) = \frac{q^2}{2}\, |x_0|^{-3} \int\limits_{K} \frac{\operatorname{sign}_{\mathrm{p}}(\mathrm{p} - \mathrm{p}x^2)}{|1 - x^2|\,|x^2|}\, dx - \frac{q^2}{2} \int\limits_{|x| > 1} \frac{dx}{|x|^4}. \tag{3}$$

The second integral is easily computed:

$$\int\limits_{|x| > 1} \frac{dx}{|x|^4} = \sum_{k=1}^{\infty} \int\limits_{|x| = q^k} \frac{dx}{|x|^4} = \left(1 - \frac{1}{q}\right) \sum_{k=1}^{\infty} q^{-3k} = \frac{1}{q(q^2 + q + 1)}$$

Now we turn to the evaluation of the first integral in (3).

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† The transformation $t = \dfrac{1 + \sqrt{\mathrm{p}}x}{1 - \sqrt{\mathrm{p}}x}$ is an analogue of the Cayley transformation for the field of real numbers. Note that when x ranges over the domain $|x| > 1$, then t ranges over another component of the circle $t\bar{t} = 1 : |1 + t| < 1$.

The normalizing factor $\frac{1}{2}$ in the formula for the measure is explained by the fact that $\int\limits_{|x| \leq 1} dx = 1$, whereas $\int\limits_{t\bar{t}=1, |1-t|<1} d^*t = \frac{1}{2}$.

First, we split it into three terms:

$$I = \int_K \frac{\operatorname{sign}_\mathfrak{p}(\mathfrak{p} - \mathfrak{p}x^2)}{|1 - x^2|\,|x^2|}\,dx = \int_{|x|>1} \frac{\operatorname{sign}_\mathfrak{p}(\mathfrak{p} - \mathfrak{p}x^2)}{|1 - x^2|\,|x^2|}\,dx$$
$$+ \int_{|x|<1} \frac{\operatorname{sign}_\mathfrak{p}(\mathfrak{p} - \mathfrak{p}x^2)}{|1 - x^2|\,|x^2|}\,dx + \int_{|x|=1} \frac{\operatorname{sign}_\mathfrak{p}(\mathfrak{p} - \mathfrak{p}x^2)}{|1 - x^2|\,|x^2|}\,dx. \tag{4}$$

We get†

$$I = \int_{|x>1|} \frac{dx}{|x|^4} + \operatorname{sign}_\mathfrak{p} \mathfrak{p} \int_{|x|<1} \frac{dx}{|x|^2} + \operatorname{sign}_\mathfrak{p} \mathfrak{p} \int_{|x|=1} \frac{\operatorname{sign}_\mathfrak{p}(1 - x^2)}{|1 - x^2|}\,dx$$
$$= \frac{1}{q(q^2 + q + 1)} - \left(\frac{-1}{q}\right) + \left(\frac{-1}{q}\right) \int_{|x|=1} \frac{\operatorname{sign}_\mathfrak{p}(1 - x^2)}{|1 - x^2|}\,dx. \tag{5}$$

The last integral can be computed by splitting the set of elements x, $|x| = 1$, into residue classes modulo $\mathfrak{p}$. We find

$$\int_{|x|=1} \frac{\operatorname{sign}_\mathfrak{p}(1 - x^2)}{|1 - x^2|}\,dx = \frac{1}{q} \sum_{a \neq 0, \pm 1} \left(\frac{1 - a^2}{q}\right)$$
$$+ \int_{|x|=1, |1-x|<1} \frac{\operatorname{sign}_\mathfrak{p}(1 - x^2)}{|1 - x^2|}\,dx + \int_{|x|=1, |1+x|<1} \frac{\operatorname{sign}_\mathfrak{p}(1 - x^2)}{|1 - x^2|}\,dx. \tag{6}$$

Here the sum is taken over the elements a of the finite field O/P of order q, other than 0 and ± 1. A straightforward calculation shows that

$$\frac{1}{q} \sum_{a \neq 0, \pm 1} \left(\frac{1 - a^2}{q}\right) = -\frac{1}{q}\left[1 + \left(\frac{-1}{q}\right)\right].$$

On the other hand, it is easy to show that each of the integrals in (6) is zero. So we find

$$I = \frac{1}{q(q^2 + q + 1)} - \left(\frac{-1}{q}\right) - \frac{1}{q}\left[1 + \left(\frac{-1}{q}\right)\right].$$

Substituting this expression in formula (3) for $I_\mathfrak{p}^{(1)}(\lambda)$ we obtain finally

$$I_\mathfrak{p}^{(1)}(\lambda) = -\frac{q}{2}\left(\frac{q^2 + q}{q^2 + q + 1} + \left(\frac{-1}{q}\right)(q + 1)\right)|x_0|^{-3} - \frac{q}{2(q^2 + q + 1)}.$$

To reach precise agreement of this formula with that of the table (p. 213, case b) we observe that

$$x_0 = \frac{1}{\sqrt{\mathfrak{p}}}\frac{\lambda - 1}{\lambda + 1}, \quad \text{and therefore} \quad |x_0| = q^{1/2}\,|\lambda - 1| = q^{1/2}\,|\lambda + \lambda^{-1} - 2|^{1/2}.$$

† The integral $\int_{|x|<1} |x|^{-2}\,dx$ is to be understood here in the sense of the regularizing value, as the value of the analytic function of s, $\varphi(s) = \int_{|x|<1} |x|^{-s}\,dx$ for $s = 2$; $\left(\frac{a}{q}\right)$ is the Legendre symbol (see p. 213).

4. Computation of the Constant c in the Inversion Formula. To obtain the value of the constant c in the inversion formula of **§ 6.2** we apply this formula to an arbitrary fixed function $f(g)$ on G.

Let U be the subgroup of matrices of G whose elements are integers of $\mathbf{K}$. Obviously U is compact and is an open set in G.

We take the function $f(g)$ that is equal to unity on U and to zero outside U, and apply the inversion formula to it.

It can be shown that $\operatorname{Tr} T_\pi(f) \neq 0$ only for the representations of the continuous series that correspond to the character $\pi(t) = |t|^{i\rho}$. Consequently, in the inversion formula for $f(g)$ only the terms corresponding to these representations occur. As a result we find

$$c = \int_U \int \mu(\pi_\rho) \operatorname{Tr} T_{\pi_\rho}(g) \, d\pi_\rho \, dg, \tag{1}$$

where $\pi_\rho(g) = |t|^{i\rho}$,

$$\mu(\pi_\rho) = -\int_K |t|^{i\rho} |1 - t|^{-2} \, dt, \tag{2}$$

Now we evaluate the integral (1). We recall that

$$\operatorname{Tr} T_{\pi_\rho}(g) = \theta(g) \frac{|\lambda_g|^{i\rho} + |\lambda_g|^{-i\rho}}{|\lambda_g - \lambda_g^{-1}|},$$

where λ_g and λ_g^{-1} are the eigenvalues of g; $\theta(g) = 1$ when $\lambda_g, \lambda_g^{-1} \in K$, $\theta(g) = 0$ otherwise. Since $|\lambda_g| = 1$ for matrices g belonging to the compact subgroup U, we have

$$\int_U \operatorname{Tr} T_{\pi_\rho}(g) \, dg = 2 \int_U \theta(g) \, |\lambda_g - \lambda_g^{-1}|^{-1} \, dg.$$

So we see that this integral does not depend on π_ρ. Therefore,

$$c = -2 \int_U \theta(g) \, |\lambda_g - \lambda_g^{-1}|^{-1} \, dg \int_K |t|^{i\rho} |1 - t|^{-2} \, dt \, d\pi_\rho. \tag{3}$$

The second integral in (3) is easily computed:†

$$\int_K |t|^{i\rho} |1 - t|^{-2} \, dt \, d\pi_\rho = \frac{q}{q-1} \int_{|t|=1} |1 - t|^{-2} \, dt = -\frac{2}{q-1} * .$$

We do not evaluate here the first integral, but give only the final result:‡

$$\int_U \theta(g) \, |\lambda_g - \lambda_g^{-1}|^{-1} \, dg = \frac{(q+1)(q-1)}{2q^2}.$$

† The factor $q(q-1)^{-1}$ arises as a consequence of the chosen normalization of $d\pi_\rho$. For we postulate that

$$\int \pi_\rho(t) \, |t|^{-1} \, dt \, d\pi_\rho = 1$$

(see **§ 2.9**).

‡ This integral may be computed by representing the matrix g of U in the form $g = z^{-1} \delta \zeta z$, where $z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$, $\delta = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$, $\zeta = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$, and taking the elements z, ζ, and λ as parameters of g. It then turns out that $dg = |\lambda_g - \lambda_g^{-1}| \, d\lambda \, d\zeta \, dz$.

And so we have the final value of the constant c:

$$c = \frac{2(q+1)}{q^2}.$$

5. The Inversion Formula for Connected Fields. We now consider the case of a connected field $\mathbf{K}$, that is, the field of complex or of real numbers. It can be shown that then the inversion formula is similar to that for a disconnected field.

If G is the group of complex matrices, then the inversion formula takes the form

$$\delta(g) = \int \mu(\pi) \operatorname{Tr} T_\pi(g)\, d\pi, \tag{1}$$

where

$$\mu(\pi) = c \int \pi(t)\, |1 - t|^{-2}\, dt. \tag{2}$$

The integration in (2) is taken over the complex plane of t; the integral (2) must be understood in the sense of the regularizing value (see **§ 2.9**).†

If G is the group of real matrices, then the inversion formula takes the form

$$\delta(g) = \int \mu(\pi) \operatorname{Tr} T_\pi(g)\, d\pi + \sum_n \mu(\pi_n) \operatorname{Tr} T_{\pi_n}(g), \tag{3}$$

where

$$\mu(\pi) = c \int_{-\infty}^{\infty} \pi(t)\, |1 - t|^{-2}\, dt. \tag{4}$$

$$\mu(\pi_n) = c \int_{t\bar{t}=1} \pi_n(t)\, |1 - t|^{-2}\, d^*t. \tag{5}$$

Here $\pi(t)$ are the multiplicative characters of the group of real numbers, $T_\pi(g)$ are the corresponding representations of the continuous series; $\pi_n(t)$ are the characters of the group of rotations of a circle, $T_{\pi_n}(g)$ the corresponding representations of the discrete series; d^*t is the measure on the circle $t\bar{t} = 1$, normalized by the condition $\int d^*t = 1$.

The inversion formulae (1) and (3) can be derived just as in the case of a disconnected field because the integrals in the formulae can be evaluated explicitly. We have to verify that the expression $I(g)$ on the right-hand side of (1) or (3), respectively, is a function concentrated at $g = e$. After that it is not hard to show that $I(g) = c\delta(g)$. We omit the detailed derivation of (1) and (3).

† We recall that in our notation $|z|$ denotes the *square* of the modulus of the complex number z.

Note that the computation of the integral for $\mu(\pi)$ leads to essentially distinct expressions for the field of complex and the field of real numbers. In fact, for the field of complex numbers, every character $\pi(t)$ is of the form

$$\pi(t) = t^{n+i\rho/2}\bar{t}^{-n+i\rho/2},$$

where n is an integer and ρ a real number. Evaluating the integral (2) we find

$$\mu(\pi) = c(\rho^2 + n^2).$$

Now we take the field of real numbers. There are two types of multiplicative characters on the real line: the characters

$$\pi(t) = |t|^{i\rho},$$

where ρ is a real number, and the characters

$$\pi(t) = |t|^{i\rho} \operatorname{sign} t.$$

Evaluating the integral (4) we find that

$$\mu(\pi) = c\pi\rho \tanh \frac{\pi\rho}{2}$$

for a character of the first type, and

$$\mu(\pi) = c\pi\rho \coth \frac{\pi\rho}{2}$$

for a character of the second type.

On the circle $t\bar{t} = 1$ the characters $\pi_n(t)$ have the form

$$\pi_n(t) = e^{in \arg t}.$$

Computing the integral (5) we find that

$$\mu(\pi_n) = c\,|n|.$$

APPENDIX TO CHAPTER 2

1. Some Facts from the Theory of Operator Rings in Hilbert Space. Here we confine ourselves to the statement of results. Their proofs can be found, for example, in Dixmier [14] or Naimark, [52].

A von Neumann algebra is a ring R of operators in Hilbert space satisfying the following conditions:

1. R contains the identity operator;
2. If $A \in R$, then $A^* \in R$, where A^* is the operator adjoint to A;

3. R is closed in the weak operator topology.

For every set S of operators in Hilbert space we denote the collection of all operators that commute with the operators from S by S'. It is easy to verify that when S contains with each operator its conjugate operator, then S' is a von Neumann algebra. If the original set S is a von Neumann algebra, then $(S')' = S$.

A von Neumann algebra R is called a *factor* if $R \cap R'$ consists only of scalar operators. Every von Neumann algebra can be realized canonically as a direct sum (possibly continuous) of factors.

If H is a finite-dimensional Hilbert space, then all factors may be obtained by the following construction. We represent H as a tensor product of two spaces H_1 and H_2: $H = H_1 \otimes H_2$. For R we take the set of all operators of the form $A \otimes 1$. Then R' consists of the operators of the form $1 \otimes B$, and the intersection $R \cap R'$ obviously contains only scalar operators. Of course, this construction is also applicable to infinite-dimensional spaces. But in an infinite-dimensional space not all the factors can be obtained in this way. Those that can be obtained are called factors of type I.

It is customary to classify factors by the structure of the set of projection operators in the factor. Factors of type I are characterized by the property that they are all minimal projections in this set (corresponding to operators of the form $P \otimes 1$, where P is a projection operator of rank 1).

In factors of type II there are no minimal projections, but there are so-called finite projections, that is, projections that are not adjoint to their regular part.

In factors of type III there are neither minimal, nor finite projections.

A representation T of a group G is called a factor-representation if the ring generated by all the operators $T(g)$, $g \in G$, is a factor. We say that a group G belongs to type I if each factor representation of it is generated by a factor of type I and is, therefore, a multiple of an irreducible representation.

Let G_1 and G_2 be two groups and T an irreducible representation of their direct product $G = G_1 \cdot G_2$. We denote by R_i the ring generated by the operators $T(g), g \in G_i \subset G$. Then $R_1' \cap R_2' = \{\lambda E\}$ by the irreducibility of T. Moreover, $R_1 \subset R_2'$, because the elements of G_1 and G_2 commute. Hence, it follows that

$$R_1 \cap R_1' \subset R_2' \cap R_1' = \{\lambda E\}$$

so that R_1 is a factor. The same is true for R_2.

If at least one of the groups G_1 and G_2 is of type I, then the restriction of T to this group is a multiple of an irreducible representation. In this case it is easy to show that the representation T is of the form $T_1 \otimes T_2$, where T_i are representations of G_i.

In the general case this is not true. One of the simplest examples can be constructed as follows. Let G be a countable discrete group in which each class of conjugate elements, except the unit class, is infinite. (An example of such a group is the group of rational matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, or the group of permutations of a countable set that shift only a finite number of points.) We consider the representation T of $G \cdot G$ in the space $L^2(G)$ given by the formula: $T(g_1, g_2)f(g)) = f(g_1^{-1}gg_2)$. This representation is irreducible, but cannot be written in the form $T_1 \otimes T_2$, where the T_i are representations of G. The restriction of T to G is not a multiple of an irreducible representation and is a factor of type II.

We now assume that the irreducible representation T of $G = G_1 \cdot G_2$ has the following property.

There exists a function $\varphi \in L_1(G_1 \cdot G_2)$ of the form

$$\varphi(g_1, g_2) = \varphi_1(g_1)\varphi_2(g_2)$$

for which

$$T(\varphi) = \int \varphi(g_1, g_2) T(g_1, g_2)\, dg_1\, dg_2$$

is a nonzero completely continuous operator. We show that then T is a tensor product of irreducible representations of G_1 and G_2.

First we observe that when φ satisfies condition (A), so does the function† $\psi = \varphi * \varphi^*$. It also is of the form $\psi_1(g_1)\psi_2(g_2)$, where $\psi_i = \varphi * \varphi_i^*$. The operators $A_i = \int \psi_i(g) T(g)\, dg$ are nonnegative, commute with each other, and their product is completely continuous. Hence, it is easy to deduce that each operator A_i has a pure point spectrum. Furthermore, if H_i is an eigenspace for A_i corresponding to a nonzero eigenvalue, then the intersection $H_1 \cap H_2$ is finite-dimensional, because all the vectors in this intersection are eigenvectors for $A_1 A_2$ with nonzero eigenvalues. The projection operator P_i onto H_i belongs to the factor R_i generated by the operators $T(g)$, $g \in G_i$. It is well known (see, for example, Naimark [52], Chapter 2, **§ 3**) that every factor R has the following property: if the operators X and Y lie in R and R', respectively, then the product XY is zero if and only if one of the factors is zero.

Among all the nonzero projection operators in R_2 we now consider an operator P for which the rank of the product $P_1 P$ takes the smallest value. (The existence of such an operator P is guaranteed by the fact that the rank of $P_1 P_2$ is finite.) We show that P is a minimal projector in R_2. For if P can be represented in the form

† We use the standard notation for the operators of multiplication and involution in the group ring of G. See, for example, Naimark [52].

$P' + P''$, where P' and P'' are orthogonal projectors in R_2, then at least one of the operators P_1P' or P_1P'' has a rank less than P_1P, which is impossible. Hence, R_2 is a factor of type I. As we have seen above, this implies that T is of the form $T_1 \otimes T_2$, where T_i are irreducible representations of G_i.

2. The Connection Between the Unitary Representations of the Group $\tilde{G}$ of all Nonsingular Matrices of Order 2 and the Subgroup of Matrices of the Form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. In this and the next subsections we establish some properties of irreducible unitary representations of the group of matrices of order 2 with elements from a disconnected topological field $\mathbf{K}$. It is convenient to consider instead of the group G of unimodular matrices the group $\tilde{G}$ of all nonsingular matrices. The transition from $\tilde{G}$ to G proceeds without difficulty (see **5**).

In $\tilde{G}$ we consider the subgroup G_0 of matrices of the form $g_{a.b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Our object is to prove the following proposition.

THEOREM 1. *Every unitary irreducible representation $T(g)$ of G remains irreducible upon restriction to G_0.*

We enumerate the unitary irreducible representations of G_0. All but one are one-dimensional and are of the form $V(g_{a,b}) = \pi(a)$, where π is a multiplicative character of K. The only infinite-dimensional irreducible representation is realized in the space $L^2(\mathbf{K}^*, d^*x)$ and is of the form

$$U(g_{a,b})\,\varphi(x) = \chi(bx)\,\varphi(ax),$$

where χ is a fixed nontrivial additive character of K.

The proof of the fact that there are no irreducible unitary representations of G_0 except the ones listed above proceeds by a standard device of the theory of induced representations, and we omit it.

LEMMA. *The restriction of T to G_0 is a multiple of an irreducible representation.*

Proof. The restriction of T to G_0, like every unitary representation, may be realized in the form of a direct integral of irreducible representations.

First we assume that in this expansion the one-dimensional representations form a set of positive measure. Since the elements of the subgroup $N = \{g_{1,b}\}$ map into the unit operator under one-dimensional representations, the space H of T contains a vector ξ that is invariant relative to $T(g)$, $g \in N$. We assume that $\| \xi \| = 1$.

We consider the function $F_\xi(g) = (T(g)\,\xi, \xi)$. Clearly, this is a continuous positive definite function on G, and constant on the

double cosets of N. Since the matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and $\begin{pmatrix}\alpha\delta-\beta\gamma & 0\\ \gamma & 1\end{pmatrix}$ for $\gamma \neq 0$, lie in the same coset, we find

$$F_\xi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right) = F_\xi\left(\begin{pmatrix}\alpha\delta-\beta\gamma & 0\\ \gamma & 1\end{pmatrix}\right).$$

By passing to the limit, as $\gamma \to 0$, we obtain

$$F_\xi\left(\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\right) = F_\xi\left(\begin{pmatrix}\alpha\delta & 0\\ 0 & 1\end{pmatrix}\right).$$

In particular, for $\delta = \alpha^{-1}$ we have $F_\xi\left(\begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix}\right) = 1$. Hence, it follows that the vector ξ is invariant relative to the subgroup K of matrices of the form $g = \begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix}$. In fact, for such matrices g we can write

$$\|T(g)\xi - \xi\|^2 = (T(g)\xi - \xi, T(g)\xi - \xi) = 2 - 2\,\mathrm{Re}\,F_\xi(g) = 0$$

But then the function $F_\xi(g)$ must be constant on the double cosets of K. It is easy to check that for $\gamma \neq 0$, $x \neq 0$, the matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and $\begin{pmatrix}\alpha\delta-\beta\gamma & 0\\ \gamma x^2 & 1\end{pmatrix}$ lie in the same coset. Therefore,

$$F_\xi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right) = F_\xi\left(\begin{pmatrix}\alpha\delta-\beta\gamma & 0\\ \gamma x^2 & 1\end{pmatrix}\right).$$

Passing to the limit, as $x \to 0$, we find

$$F_\xi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right) = F_\xi\left(\begin{pmatrix}\alpha\delta-\beta\gamma & 0\\ 0 & 1\end{pmatrix}\right).$$

In particular, for every unimodular matrix g we have $F_\xi(g) = 1$. As above, it now follows that $T(g)\xi = \xi$ for every unimodular matrix g.

We denote by H_0 the subspace of H that consists of the vectors invariant under the unimodular subgroup. A simple computation shows that H_0 is invariant under all the operators $T(g)$, $g \in \tilde{G}$. Since H is irreducible, we must have $H_0 = H$. So we see that T is trivial on the subgroup G of unimodular matrices and can therefore be regarded as a representation of the factor group $\tilde{G}/G$. Since this is a commutative group, T must be one-dimensional. But for one-dimensional representations our lemma and Theorem 1 are trivially true.

Now we consider the second case, when there are no one-

dimensional representations of G in the expansion of T. Since G_0 has only a single representation U that is not one-dimensional, in this case the restriction of T to G_0 is a multiple of U, and the lemma is proved.

From these arguments we can derive the following more general proposition:

If T is a factor representation of $\tilde{G}$, then its restriction to G_0 is a multiple of an irreducible representation.

For the only place in our arguments where we have used the irreducibility of T is the proof of the equation $H_0 = H$. For a factor representation this equation can be proved as follows. We denote by P the projection operator onto H_0. As we have mentioned above, P commutes with all the operators $T(g)$ and hence with all the operators from the weakly closed ring R generated by $T(g)$. It is not hard to verify that P also commutes with all the operators of the ring R' consisting of the operators that commute with the elements from R. Therefore $P \in R \cap R'$. But by definition of a factor representation the rings R and R' are factors, that is, the intersection $R \cap R'$ consists only of scalar operators. Therefore $P = E$ and $H_0 = H$.

It is advantageous to go over to another realization of the representation U, by considering instead of functions on $\mathbf{K}^*$ their Fourier transforms on the dual group Π.

In this realization the representation operators take the form

$$U(g_{a,b})\,\varphi(\pi_1) = \int \pi_1\pi_2^{-1}(b)\,\Gamma(\pi_2\pi_1^{-1})\,\pi_2(a)\,\varphi(\pi_2)\,d\pi_2$$

for $b \neq 0$, and

$$U(g_{a,0})\,\varphi(\pi) = \pi(a)\,\varphi(\pi). \tag{1}$$

From the lemma it follows that the restriction of T to G_0 is given by the same formulae, only instead of ordinary functions we have to consider vector functions with values in a certain Hilbert space L.

Furthermore, when g lies in the center of $\tilde{G}$, the operators $T(g)$ commute with all the representation operators and are therefore multiples of the unit operator. Hence, it follows that if $d_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, then $T(d_\lambda) = \pi_0(\lambda)E$, where π_0 is a fixed character on $\mathbf{K}^*$. We denote by s the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From the identity $s g_{a,0} s^{-1} = g_{a^{-1},0}\, d_a$ it follows that $T(s)$ is of the form

$$T(s)\,\varphi(\pi) = s(\pi)\,\varphi(\pi_0\pi^{-1}),$$

where $s(\pi)$ is a function on Π whose values are unitary operators in L.

Now we observe that the irreducibility of $T(g)$ implies that of the set of operators $s(\pi)$ in L. For if L_1 is a subspace of L invariant under all (or even almost all) $s(\pi)$, then the subspace $H_1 \subset H$

consisting of the vector functions with values in L_1 is invariant under the $T(g)$, $g \in G_0$, and under $T(s)$. But the subgroup G_0 and the element s generate the whole group $\tilde{G}$. Therefore H_1 is invariant under all the $T(g)$, and this contradicts the irreducibility of H.

To prove Theorem 1 it is now sufficient to show that the operators $s(\pi)$ commute with each other. Then the set $s(\pi)$ is irreducible only when L is one-dimensional, and so the restriction of $T(g)$ to G_0 coincides with U. The identity $sg_{1,1}s = g_{1,-1}sg_{1,-1}$ reduces to the following condition on $s(\pi)$:

$$s(\pi_1)\,\Gamma(\pi_1\pi_2\pi_0^{-1})s(\pi_2) = \pi_1(-1)\pi_2(-1)\int \Gamma(\pi\pi_1^{-1})s(\pi)\,\Gamma(\pi\pi_2^{-1})\,d\pi, \quad (2)$$

from which it follows immediately that $s(\pi_1)$ and $s(\pi_2)$ commute for almost all pairs (π_1, π_2). The proof of Theorem 1 is now complete.

3. Theorem on the Complete Continuity of the Operator T_φ. Here we show that the group $\tilde{G}$ is of type I so that all unitary factor representations of $\tilde{G}$ are multiples of an irreducible representation.

For this purpose we prove the following stronger proposition.

THEOREM 2. *If $T(g)$ is an irreducible unitary representation of $\tilde{G}$, and φ a summable function on $\tilde{G}$, then*

$$T\varphi = \int \varphi(g)\,T(g)\,dg$$

is a completely continuous operator.†

Proof. Let $\varphi'_{k,\theta}$ be the generalized function on $\tilde{G}$ given by the formula $(\varphi'_{k,\theta}, f) = q^k \int f(g_{a,b})\theta^{-1}(a)\,d^*a\,db$, where θ is a multiplicative character and the integral is taken over the set

$$g_{a,b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad |a| = 1, \quad |b| \le q^{-k}.$$

We set $\varphi_{k,\theta} = \varphi'_{k,\theta} - \varphi'_{k-1,\theta}$. It is not hard to check that $U_{\varphi_{k,\theta}}$ is a projection operator on the one-dimensional subspace of $L^2(\mathbf{K}^*)$ generated by the functions

$$e_{k,\theta}(x) = \begin{cases} \theta(y) & \text{for } |x| = q^{-k}, \\ 0 & \text{for } |x| \ne q^{-k}. \end{cases}$$

Obviously, the set of functions $e_{k,\theta}$ forms an orthogonal basis of $L^2(\mathbf{K}^*)$.

Now let M be the set of all functions $\varphi \in L^1(\tilde{G})$ for which the

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† Groups for which this statement is true are called CCR-groups, following Kaplansky who first singled out this class of groups and proved that they are of type I.

operator $T\varphi$ has finite rank. Clearly, M is a two-sided ideal in $L^1(\tilde{G})$ and contains all functions of the form $\varphi_{k,\theta} * f$, $f \in L^1(\tilde{G})$. If $u \in L^\infty(\tilde{G})$ is a functional on $L^1(\tilde{G})$ that is zero on M, then the function u and all its translates have the property $u * \varphi_{k,\theta} = 0$. Hence, $u = \text{const}$ and the closure of M contains all the functions on $L^1(\tilde{G})$ whose integral is zero.

On the other hand, there are functions on M with a nonzero integral, for example the characteristic function of U (see the next subsection). Therefore, $\bar{M} = L^1(\tilde{G})$. So we have shown that every function $\varphi \in L^1(\tilde{G})$ may be approximated in the norm of $L^1(\tilde{G})$ by functions from M. Hence, the operator $T\varphi$ may be approximated (in the sense of the topology defined by the operator norm) by operators of finite rank and is, therefore, completely continuous. The proof of the theorem is now complete.

4. The Decomposition of an Irreducible Representation of $\tilde{G}$ Relative to Representations of its Maximal Compact Subgroup. The Theorem on the Existence of a Trace. Our object is to prove the following proposition.

THEOREM 3. *Let $T(g)$ be a unitary irreducible representation of $\tilde{G}$. In the decomposition of the restriction of $T(g)$ to a maximal compact subgroup $U \subset \tilde{G}$ every irreducible component has finite multiplicity.*

Proof. We are going to use results obtained in the proof of Theorem 1. We have seen that the representation operator $T(s)$ corresponding to the matrix $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has the following form in the π-realization:

$$T(s)\,\varphi(\pi) = s(\pi)\,\varphi(\pi_0\pi^{-1}), \tag{1}$$

where π ranges over the character group Π dual to $\mathbf{K}^*$. Here the function $s(\pi)$ satisfies the following relation:

$$s(\pi_1)\,\Gamma(\pi_1\pi_2\pi_0^{-1})s(\pi_2) = \pi_1\pi_2(-1)\int \Gamma(\pi\pi_1^{-1})s(\pi)\,\Gamma(\pi\pi_2^{-1})\,d\pi. \tag{2}$$

We expand $s(\pi)$ in a Laurent series and find the relations for the coefficients of the expansions.

We recall that according to **§ 2.6** every character is given by a complex number λ, $|\lambda| = 1$, and a character $\theta(y)$ defined on the group O^* of elements of norm 1. It is expressed by the following formula: if $x = \mathfrak{p}^k y$, $|y| = 1$, then

$$\pi(x) = \lambda^k\theta(y).$$

We substitute in (2) the expansions of $s(\pi) = s(\lambda, \theta)$ and of $\Gamma(\pi)$ in Laurent series

Then we obtain the following relation for the coefficients $s_k(\theta)$:†

$$(1 - q^{-1}) \sum_m s_{k+m}(\theta_1)\, \Gamma_{-m}(\theta_1\theta_2\theta_0^{-1})\, \lambda_0^m s_{l+m}(\theta_2) = \theta_1\theta_2(-1) \sum_\theta \Gamma_{-k}(\theta\theta_1^{-1}) s_{k+l}(\theta)\, \Gamma_{-l}(\theta\theta_2^{-1}). \quad (3)$$

We investigate this condition by taking account of the following formulae for the coefficients of $\Gamma_k(\theta)$ obtained in **§ 2.6.**

If the rank of θ is $m > 0$, then $\Gamma_k(\theta) = 0$ for all $k \neq -m$;

$$|\Gamma_{-m}(\theta)| = q^{-m/2}.$$

If the rank of θ is 0, that is, $\theta(x) \equiv 1$, then

$$\Gamma_k(\theta) = \begin{cases} q^{-1} & \text{for } k < -1, \\ -q^{-1} & \text{for } k = -1, \\ 1 - q^{-1} & \text{for } k > -1. \end{cases}$$

First of all, by taking (3) for fixed θ_1, θ_2, l and sufficiently large positive k, we see that the right-hand side is 0, and for $\theta_1, \theta_2 \neq \theta_0$ the sum on the left-hand side reduces to a single term in which m is the rank of $\theta_1\theta_2\theta_0^{-1}$.‡ Hence, it is easy to derive that for each θ the coefficients $s_k(\theta)$ vanish for sufficiently large positive k.

Secondly, if $k \leq 0$, $l \leq 0$, $\theta_1 \neq \theta_2$, $\theta_1\theta_2 \neq \theta_0$, then it follows from (3) that $s_{k+m}(\theta_1)s_{l+m}(\theta_2) = 0$, where m is the rank of the character $\theta_1\theta_2\theta_0^{-1}$. We assume that for some θ_1 and some $n \leq 0$ the coefficient $s_n(\theta_1)$ is different from zero. Setting $k = n - m$ we then find that for all θ_2, other than θ_1 and $\theta_0\theta_1^{-1}$, the coefficients $s_{l+m}(\theta_2)$ are zero for $l \leq 0$. So we have shown that for all θ, except possibly the one pair θ_1, $\theta_0\theta_1^{-1}$, among the coefficients $s_k(\theta)$ there are only finitely many different from zero. Finally, for the excluded characters θ_1 and $\theta_0\theta_1^{-1}$ we easily obtain from (3) a recurrence relation between the $s_k(\theta)$ from which it follows that $|s_k(\theta)|$ decreases like a geometric progression as $k \to -\infty$ (see **6**).

From all we have shown it follows that *the function* $s(\pi) = s(\lambda, \theta)$ *is infinitely differentiable with respect to* λ *for every* θ.

Now we are in a position to prove Theorem 3. We note first that all the maximal compact subgroups of $\tilde{G}$ are conjugate to the group U consisting of those matrices g for which the matrix elements of g and of g^{-1} belong to O. U has a family of normal subgroups U_n, consisting of the matrices that are congruent to the unit matrix modulo p^n.

Obviously, U itself as well as the subgroups U_n are open subsets of $\tilde{G}$ and form a complete system of neighborhoods of the unit element of $\tilde{G}$.

† Throughout we denote by (λ_0, θ_0) the components of the character π_0.

‡ For the definition of rank see **§ 2.6.**

It is easy to check that every irreducible representation of U is trivial on U_n for sufficiently large n.

We denote by H_n the subspace of H consisting of the vectors that are invariant under the operators $T(g)$, $g \in U_n$. Theorem 3 is equivalent to the statement that all these spaces H_n are finite-dimensional.

First, we find the space $H_n^0 \subset H_n$ consisting of the vectors that are invariant under the $T(g)$, $g \in U_n \cap G_0$. This is very easy if we use the original realization of the representation of U. We only state the final result.

The space H_n^0 consists of the functions $\varphi(\pi) = \sum \varphi_k(\theta)\lambda^k$, satisfying the condition $\varphi_k(\theta) = 0$ if (rank θ) $> n$ or $k < -n$.

Since $sU_ns^{-1} = U_n$, H_n is invariant under $T(s)$. Therefore, if $\varphi(\pi) \in H_n$, the functions $\varphi(\pi)$ and $T(s)\varphi(\pi) = s(\pi)\varphi(\pi_0\pi^{-1})$ both satisfy the conditions stated above, or both fail to do so. The fact that H_n is finite-dimensional now follows from the infinite differentiability with respect to λ of the function $s(\pi) = s(\lambda, \theta)$ and from the following easily verified proposition.

Let $s(\lambda)$ be an infinitely differentiable function on the unit circle Λ and $|s(\lambda)| \equiv 1$. Then the space $L^2(\Lambda)$ contains only a finite number of linearly independent functions $\varphi(\lambda)$ satisfying the conditions:

1. *The function $\varphi(\lambda)$ is orthogonal to λ^k for $k < -n$.*
2. *The function $s(\lambda)\varphi(\lambda)$ is orthogonal to λ^k for $k > n$.*

The proof of the theorem is now complete.

Observe that in the proof of Theorem 3 we have nowhere used the maximality of the compact subgroup. In fact, it is sufficient to postulate that the compact subgroup U contains all the subgroups U_n for sufficiently large n, in other words, that it is an open compact subgroup of $\tilde{G}$. Thus, Theorem 3 remains valid for every open compact subgroup U of $\tilde{G}$, in particular, for each of the subgroups U_n.

COROLLARY. *If φ belongs to the space S of finite piecewise constant functions on $\tilde{G}$, then the operator $T\varphi = \int \varphi(g)T(g)\,dg$ has finite rank.*

Proof. For each function φ in S we can find an n such that φ is constant on the double cosets of U_n. Hence, it follows that for every x in the representation space H and every $g \in U_n$ we have

$$T(g)T_\varphi = T_\varphi.$$

So we see that the domain of values of the operator $T\varphi$ lies in the space H_n consisting of the vectors that are invariant under U_n. The fact that H_n is finite-dimensional was established in the proof of Theorem 3.

From what we have shown it follows that the operator T_φ has a trace, and that this trace is a linear functional in S. This fact can also be stated in the following way:

For every irreducible unitary representation $T(g)$ of $\tilde{G}$ whatever the trace Tr $T(g)$ *of the operator $T(g)$ exists as a generalized function in the space S.*

5. Representations of the Unimodular Group. Let us show how we can carry over the Theorems 2 and 3, which we have previously proved for the full matrix group $\tilde{G}$, to the group G of matrices with determinant 1. Here we confine ourselves to the case when the group $W = \mathbf{K}^*/(\mathbf{K}^*)^2$ is finite. In **§ 1** it was shown that if the finite field $L = O/P$ has characteristic other than 2, then W is of order 4.

If the field L has characteristic 2 and $\mathbf{K}$ characteristic 0 (as, for example, in the important case of the field of two-adic numbers), then G is also finite. For the series

$$(1 - x)^{1/2} = 1 - \sum_{n=1}^{\infty} \frac{(2n - 1)!!}{(2n)!!} x^n$$

converges for $\left|\frac{x}{4}\right| < 1$. Therefore $(\mathbf{K}^*)^2$ contains O_n^* for sufficiently large n. Furthermore, all the even powers of the generator p occur in $(\mathbf{K}^*)^2$. Hence, the order of W does not exceed the order of the finite group $Z_2 \cdot O^*/O_n^*$.

But if $\mathbf{K}$ has characteristic 2, then W is an infinite group (it is isomorphic to the product of a countable number of groups Z_2). We exclude this case from our discussion, although even here one can prove that G is of type I.

Let $\hat{G}$ be the set of all irreducible representations, identified up to equivalence, of G. The group $\tilde{G}$ acts in $\hat{G}$ in the following way. If $T \in \hat{G}$, $g \in \tilde{G}$, then we set $T^{(g)}(g_1) = T(gg_1g^{-1})$. Clearly, $T^{(g)}$ is also an irreducible unitary representation of G. We consider the stability subgroup of the point $T \in G$. Clearly, this subgroup contains G, because if $g \in G$, then $T^{(g)}(g_1) = T(g)T(g_1)T^{-1}(g)$, from which it is obvious that T and $T^{(g)}$ are equivalent. Furthermore, all the matrices of the form $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ belong to the stability group, because for such g the representation $T^{(g)}$ simply coincides with T.

Therefore the stability subgroup contains all the matrices of $\tilde{G}$ whose determinant belongs to $(\mathbf{K}^*)^2$. Since we have assumed that $\mathbf{K}^*/(\mathbf{K}^*)^2$ is finite, the orbit of T under the action of $\tilde{G}$ consists of a finite number of points $T_1, T_2, \ldots, T_k$. Let $H_1, H_2, \ldots, H_k$ be the spaces in which these representations act. Then we can give in the direct sum $H_1 \oplus H_2 \oplus \cdots \oplus H_k$ an irreducible representation T' of $\tilde{G}$ whose restriction to G leaves every H_i invariant and coincides in this subspace with T_i.

Theorems 2 and 3 for T now follow easily from the same theorems for T'.

6. Classification of all Irreducible Representations of G and $\tilde{G}$. Condition (3) in **4** enables us to give a complete classification of all irreducible unitary representations of $\tilde{G}$ and G.

THEOREM 4. *These are no irreducible unitary representations of G other than representations of the principal, the supplementary, and the discrete series, together with the special and the unit representation.*

We find it useful to prove an analogous theorem for $\tilde{G}$. We give a list of the representations of this group, where g denotes the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \tilde{G}$.

1. The continuous series consists of the representations T_{π_1,π_2} where π_1 and π_2 are unitary multiplicative characters of $\mathbf{K}$. The representation space is $L^2(\mathbf{K}, dx)$ (see **§ 3.1**), and

$$T_{\pi_1,\pi_2}(g)f(x) = \pi_1(\beta x + \delta)\pi_2\left(\frac{\alpha\delta - \beta\gamma}{\beta x + \delta}\right)\frac{|\alpha\delta - \beta\gamma|^{\frac{1}{2}}}{|\beta x + \delta|}f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right).$$

2. The supplementary series consists of the representations $V_{\pi_0,\rho}$, where π_0 is a unitary multiplicative character of $\mathbf{K}$, and ρ a real number from the interval $(0, 1)$. The representation space consists of the functions on K with the scalar product (see **§ 3.7**)

$$(f_1, f_2) = \int f_1(x)\overline{f_2(y)}\,|x - y|^{-2\rho}\,dx\,dy.$$

The representation operators act according to the formula

$$V_{\pi_0,\rho}(g)f(x) = \pi_0(\alpha\delta - \beta\gamma)\,\frac{|\alpha\delta - \beta\gamma|^{1-\rho}}{|\beta x + \delta|^{2-2\rho}}f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right).$$

3. The singular series consists of the representations S_{π_0}, where π_0 is a unitary multiplicative character of $\mathbf{K}$. The representations of this series act in the space of functions on $\mathbf{K}$ for which $\int_K f(x)\,dx = 0$, and the scalar product is given by the formula (see **§ 3.8**)

$$(f_1, f_2) = \int f_1(x)\overline{f_2(y)}\,\ln|x - y|\,dx\,dy.$$

The representation operators are of the form

$$S_{\pi_0}f(x) = \pi_0(\alpha\delta - \beta\gamma)\,\frac{|\alpha\delta - \beta\gamma|}{|\beta x + \delta|^2}f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right).$$

4. The discrete series consists of the representations $U_{\pi_0,\Pi}$, where π_0, Π are unitary multiplicative characters of $\mathbf{K}$ and $\mathbf{K}(\sqrt{\tau})$, respectively. The representation acts in the space $L^2(\mathbf{K}, dx)$ according to the formula (see **§ 4.1**)

$$U_{\pi_0,\Pi}(g)f(x)$$

$$= \begin{cases} \displaystyle\int K(g \mid x, y) f(y)\, dy, & \text{when } \beta \neq 0 \\ \pi_0(\alpha\delta) \operatorname{sign}_\tau(\alpha)\Pi(\alpha^{-1}) \left|\dfrac{\delta}{\alpha}\right|^{1/2} \chi\!\left(\dfrac{\gamma x}{\alpha}\right) \varphi\!\left(\dfrac{\delta x}{\alpha}\right), & \text{when } \beta = 0. \end{cases}$$

The kernel $K(g \mid x, y)$ is different from zero only for $\operatorname{sign}_\tau(xy\Delta) = 1$ and has the form

$$K(g \mid x, y)$$

$$= a_\tau c_\tau \pi_0(\Delta)\, |\Delta|^{1/2} \frac{\operatorname{sign}_\tau \beta x\Delta}{|\beta|} \chi\!\left(\frac{\delta x + \alpha y}{\beta}\right) \int\limits_{t\bar{t}=y/\Delta x} \chi\!\left(\frac{x\,\Delta t + yt^{-1}}{-\beta}\right) \Pi(t)\, d^*t,$$

where we have set $\Delta = \alpha\delta - \beta\gamma$.

5. The degenerate series consists of the one-dimensional representations

$$W_{\pi_0}(g) = \pi_0(\alpha\delta - \beta\gamma),$$

where π_0 is a unitary multiplicative character of K.

THEOREM 4′. *There are no irreducible unitary representations of $\tilde{G}$ other than the representations listed above.*

Proof. First we consider a finite-dimensional representation T of $\tilde{G}$. The operators $P_n = \int_{U_n} T(g)\, dg$ are obviously self-adjoint projection operators in the representation space H of T. Furthermore, since the subgroups U_n form a complete system of neighborhoods of the unit element of $\tilde{G}$, the sequence $\{P_n\}$ strongly converges to the unit operator.†

In a finite-dimensional space this is possible only if for all n after a certain n, we have $P_n = E$. But this means that all the vectors in H are invariant under U_n. The kernel of T is a normal subgroup of $\tilde{G}$ containing U_n. Hence, it follows that the kernel of T contains the whole subgroup G. Therefore, T is, in fact, a representation of the factor group $\tilde{G}/G$.

The latter group is commutative and isomorphic to the multiplicative group of **K**. Hence, the only finite-dimensional unitary irreducible representations of $\tilde{G}$ are the representations W_{π_0}, which form the degenerate series.

Now let T be an infinite-dimensional irreducible representation. As we have seen in **2**, the restriction of T to G_0 is also irreducible and coincides with a certain fixed representation of G_0 given by the formulae (1).

† We recall that U_n consists of all matrices that are congruent to E modulo $\mathfrak{p}^n O$.

Since $\tilde{G}$ is generated by G_0 and the matrix $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, T is given when we know the operator $T(s)$. This operator, as was shown in **2**, has the form

$$T(s)\,\varphi(\pi) = s(\pi)\,\varphi(\pi_0\pi^{-1}),$$

where $s(\pi)$ is a function on the set Π of unitary multiplicative characters of $\mathbf{K}$ that assumes complex values of modulus 1. The character π_0 in this formula is determined by the equation $T(d_\lambda) = \pi_0(\lambda)E$, where $d_\lambda = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$. Now we give explicit expressions for the functions $s(\pi)$ corresponding to the series of representations listed above.† The symbol $\pi_{(\rho)}$ denotes the non-unitary character $\pi_{(\rho)}(x) = |x|^\rho$.

1. $T = T_{\pi_1,\pi_2}$, a representation of the principal series

$$s(\pi) = \pi_1\pi_2(-1)\,\Gamma(\pi^{-1}\pi_1\pi_{(1/2)})\,\Gamma(\pi^{-1}\pi_2\pi_{(1/2)}).$$

2. $T = V_{\pi_0,\rho}$, a representation of the supplementary series

$$s(\pi) = \Gamma(\pi^{-1}\pi_0\pi_{(\rho)})\,\Gamma(\pi^{-1}\pi_0\pi_{(1-\rho)}).$$

3. $T = S_{\pi_0}$, a representation of the special series

$$s(\pi) = \Gamma(\pi^{-1}\pi_0)\,\Gamma(\pi^{-1}\pi^0\pi_{(1)}).$$

4. $T = U_{\pi,_0\Pi}$, a representation of the discrete series

$$s(\pi) = c\,\Gamma_\tau(\pi_\tau),$$

where Γ_τ is the Gamma-function of $\mathbf{K}(\sqrt{\tau})$ and π_τ the nonunitary multiplicative character of this field given by the formula

$$\pi(z) = \Pi^{-1}(z)\,\pi_0\pi^{-1}\pi_{(1/2)}(z\bar{z}).$$

To prove Theorem 4′ it is enough to verify that the functional equation (2) derived in **2** has no solutions other than the ones listed above. We shall do this in the following way.

First, we explicitly find all the solutions of this functional equation that have in their expansion as a Laurent series

$$s(\pi) = s(\lambda, \theta) = \sum_k s_k(\theta)\,\lambda^k \tag{1}$$

at least one nonzero coefficient for a nonpositive power of λ. It turns out that all such solutions are connected with the representations of the principal, the supplementary, and the singular series. The remaining solutions cannot be given in explicit form, but it can

† Observe that it follows immediately from these formulae that the representations T_{π_2,π_1}, and T_{π_2,π_1} are equivalent, and also the representations $V_{\pi_0,\rho}$ and $V_{\pi_0,1-\rho}$.

be shown that in the representations corresponding to these solutions the matrix elements are square-summable on G. Hence, it follows that the associated representations occur as discrete terms in the decomposition of $L^2(G)$ into irreducible components. The Plancherel formula derived in **§ 6**, which gives the decomposition of $L^2(G)$ into representations of the fundamental and the discrete series, then shows that the remaining solutions of our functional equation are connected with the representations of the discrete series.

Now we carry out this program.

We recall some results of **4.** The fundamental functional equation, in terms of the coefficients of the Laurent series, has the form

$$(1 - |\mathfrak{p}|) \sum_m s_{k+m}(\theta_1)\lambda_0^m \Gamma_{-m}\left(\frac{\theta_1\theta_2}{\theta_0}\right) s_{l+m}(\theta_2)$$
$$= \theta_1\theta_2(-1) \sum_\theta \Gamma_{-k}\left(\frac{\theta}{\theta_1}\right) s_{k+l} \Gamma_{-l}\left(\frac{\theta}{\theta_2}\right). \quad (2)$$

The coefficients $s_k(\theta)$ have the following properties. For each θ there exists a number $\rho(\theta)$ such that $s_k(\theta) = 0$ for $k > \rho(\theta)$. If only a finite number of the coefficients $s_k(\theta)$ is different from zero, then $s(\lambda, \theta) = s_{\rho(\theta)}(\theta) \cdot \lambda^{\rho(\theta)}$ (that is, only one coefficient is different from zero; this follows easily from the fact that $|s(\lambda, \theta)| = 1$ for $|\lambda| = 1$).

We use the notation $\theta^* = \theta_0\theta^{-1}$. Then $s_k(\theta^*) = s_k(\theta)$.

We call a character θ exceptional if among the coefficients $s_k(\theta)$, $k \leq 0$, there is at least one different from zero. We have shown that these are not more than two exceptional characters. We treat three cases separately.

1. There are two distinct exceptional characters θ_1 and θ_1^*. Let $s_{-n}(\theta_1) \neq 0$, $n \geq 0$. Let θ_2 be any character other than θ_1^*. We denote by r^* the rank of the character $\dfrac{\theta_1\theta_2}{\theta_0} = \dfrac{\theta_1}{\theta_1^*}$. Then the sum on the left-hand side of (2) reduces to the single term

$$s_{k+r^*}(\theta_1)\,\Gamma_{-r^*}\left(\frac{\theta_2}{\theta_1^*}\right)\lambda_0^{r^*} s_{l+r^*}(\theta_2).$$

We set $k = -n - r^*$. Then the sum on the right also reduces to a single term, because $\Gamma_{n+r^*}\left(\dfrac{\theta}{\theta_1}\right)$ is different from zero only for $\theta = \theta_1$. So we obtain the equation

$$(1 - |\mathfrak{p}|)s_{-n}(\theta_1)\,\Gamma_{-r^*}\left(\frac{\theta_2}{\theta_1}\right)\lambda_0^{r^*} s_{l+r}(\theta_2)$$

Since $\Gamma_{n+r^*}(1) = 1 - |\mathfrak{p}|$, we have

$$s_{l+r^*}(\theta_2) = \frac{\theta_1\theta_2(-1)s_{l-n-r^*}(\theta_1)\,\Gamma_{-l}\left(\dfrac{\theta_1}{\theta_2}\right)}{s_{-n}(\theta_1)\,\Gamma_{-r^*}\left(\dfrac{\theta_2}{\theta_1}\right)\lambda_0^{r^*}}. \tag{4}$$

In particular, if $\theta_2 \neq \theta_{1_\theta}$, then $s_{l+r^*}(\theta_2)$ differs from zero only when l is the rank of $\dfrac{1}{\theta_2}$, which we denote by r.

Finally, if θ is a character other than θ_1 and θ_1^*, then

$$s(\lambda, \theta) = \theta\theta_1(-1)\frac{s_{r-r^*-n}(\theta_1)\,\Gamma_{-r}\left(\dfrac{\theta_1}{\theta}\right)}{s_{-n}(\theta_1)\,\Gamma_{-r^*}\left(\dfrac{\theta}{\theta_1^*}\right)\lambda_0^{r^*}}\lambda^{r+r^*}. \tag{5}$$

Now, we set $\theta_2 = \theta_1$ in (2) and denote by r_0 the rank of $\dfrac{\theta_1}{\theta_1^*}$. If at least one of the numbers k or l is nonpositive, then the sum on the right-hand side reduces to the single term with $\theta = \theta_1$, and we obtain the equation

$$(1 - |\mathfrak{p}|)s_{k+r_0}(\theta_1)\,\lambda_0^{r_0}\,\Gamma_{-r_0}\left(\frac{\theta_1}{\theta_1^*}\right)s_{l+r_0}(\theta_1) = \Gamma_{-k}(1)s_{k+l}(\theta_1)\,\Gamma_{-l}(1).$$

After the substitution $s_k(\theta_1) = \sigma_{2r_0-k}\dfrac{1 - |\mathfrak{p}|}{\Gamma_{-r_0}\left(\dfrac{\theta_1}{\theta_1^*}\right)\lambda_0^{r_0}}$ this becomes the relation

$$\sigma_k\sigma_l = \begin{cases} \sigma_{k+l} & \text{for } k \geq r_0, \quad l \geq r_0, \\ \dfrac{|\mathfrak{p}|}{1 - |\mathfrak{p}|} & \text{for } k = r_0 - 1, \quad l \geq r_0, \\ 0 & \text{for } k < r_0 - 1, \quad l \geq r_0. \end{cases}$$

From this the quantities σ_k can easily be found. We only state the final result. There exists a complex number τ such that $\sigma_k = \dfrac{\Gamma_{k-r_0}(1)\tau^k}{1 - |\mathfrak{p}|}$. For $s_k(\theta_1)$ we now derive the expressions†

$$s_k(\theta_1) = \frac{\Gamma_{r_0-k}(1)\tau^{2r_0-k}}{\lambda_0^{r_0}\,\Gamma_{-r_0}\left(\dfrac{\theta_1}{\theta_1^*}\right)} = \theta_0(-1)\frac{\Gamma_{r_0-k}(1)\tau^{2r_0-k}\,\Gamma_{-r_0}\left(\dfrac{\theta_1^*}{\theta_1}\right)}{\lambda_0^{r_0}\,|\mathfrak{p}|^{r_0}}, \tag{6}$$

† Here, and subsequently, we use the identity $\overline{\Gamma_k(\theta)} = \theta(-1)\Gamma_k(\theta^{-1})$ (see § 2.6).

whence

$$s(\lambda, \theta_1) = \sum_k \theta_0(-1) \frac{\Gamma_{r_0-k}(1)\, \lambda^k \tau^{2r_0-k}\, \Gamma_{-r_0}\left(\frac{\theta_1^*}{\theta_1}\right)}{\lambda_0^r\, |\mathfrak{p}|^{r_0}}$$

$$= \sum_k \theta_0(-1)\, \Gamma_{r_0-k}(1) \left(\frac{\tau}{\lambda}\right)^{r_0-k} \Gamma_{-r_0}\left(\frac{\lambda_0\, |\mathfrak{p}|^{1/2}}{\tau\lambda}\right)^{-r_0}. \qquad (7)$$

Substituting the value of $s_k(1)$ from (6) in (5) we find that for θ, other than θ_1 and θ_1^*,

$$s(\lambda, \theta) = \theta\theta_1(-1) \frac{\tau^{r^*-r}\, \Gamma_{-r}\left(\frac{\theta_1}{\theta}\right) \lambda^{r+r^*}}{\Gamma_{-r^*}\left(\frac{\theta}{\theta_1^*}\right) \lambda_0^{r^*}}$$

$$= \theta_0(-1)\, \Gamma_{-r}\left(\frac{\theta_1}{\theta}\right) \left(\frac{\tau}{\lambda}\right)^{-r} \Gamma_{-r^*}\left(\frac{\theta_1^*}{\theta}\right) \left(\frac{\lambda_0\, |\mathfrak{p}|}{\tau\lambda}\right)^{-r^*}. \qquad (8)$$

Note that $|s(\lambda, \theta)| = 1$ is equivalent to $|\tau| = |\mathfrak{p}|^{1/2}$.

Now we compare these formulae (7) and (8) with the previously derived expression

$$s(\pi) = \pi_1\pi_2(-1)\, \Gamma(\pi^{-1}\pi_1\pi_{(1/2)})\, \Gamma(\pi^{-1}\pi_2\pi_{(1/2)}) \qquad (9)$$

for the function $s(\pi)$ corresponding to the representation T_{π_1,π_2}.

We denote by π_1 and π_2 the characters with the coordinates $(\tau\, |\mathfrak{p}|^{-1/2}, \theta_1)$ and $\left(\frac{\lambda_0\, |\mathfrak{p}|^{1/2}}{\tau}, \theta_1^*\right)$, respectively. It is not hard to check that for these π_1 and π_2 the formula (9), rewritten in the coordinates (λ, θ), turns into (7) and (8).

So we have shown that all the solutions of the functional equation (2) having two exceptional characters are connected with the representations of the principal series.

2. There exists only one exceptional character $\theta_1 = \theta_1^*$. We may assume that $\theta_1 \equiv 1$. For it is easy to verify that the function $\tilde{s}(\pi) = s(\pi\tilde{\pi})$ satisfies a functional equation of the form (2) in which π_0 is replaced by $\pi_0\tilde{\pi}^{-2}$. If the excluded character for $s(\pi)$ is θ, then for $\tilde{s}(\pi)$ it is $\theta\tilde{\theta}^{-1}$. The same argument shows that we may confine our discussion to the case $\lambda_0 = 1$.

After the substitution $\theta_1 \equiv 1$ the fundamental equation (2) takes the form

$$(1 - |\mathfrak{p}|) \sum_r s_{k+r}(1)\, \Gamma_{-r}(\theta_2) s_{l+r}(\theta_2)$$
$$= \theta_2(-1) \sum_\theta \Gamma_{-k}(\theta) s_{k+l}(\theta)\, \Gamma_{-l}\left(\frac{\theta}{\theta_2}\right). \qquad (10)$$

In particular, if $k \leq 0$ and the rank of θ_2 is $r_2 \neq 0$, then we obtain

$$s_{k+r_2}(1)\, \Gamma_{-r_2}(\theta_2) s_{l+r_2}(\theta_2) = \theta_2(-1) s_{k+l}(1)\, \Gamma_{-l}(\theta_2^{-1}).$$

The right-hand side of this equation can be different from zero only when $l = \text{rank of } \theta_2^{-1} = r_2$. Choosing a nonpositive k such that $s_{k+r_2}(1) \neq 0$ and setting $l = r_2$ we find

$$s_{2r_2}(\theta_2) = \frac{\theta_2(-1)\,\Gamma_{-r_2}(\theta_2^{-1})}{\Gamma_{-r_2}(\theta_2)}.$$

So we have found the coefficients $s_k(\theta)$ for $\theta \neq 1$. Now we set $k > 0$, $l = r_2$ in (10) and separate on the left-hand side the term corresponding to $\theta = 1$:

$$\begin{aligned}(1 - |\mathfrak{p}|)s_{k+r_2}(1)\,\Gamma_{-r_2}(\theta_2)s_{2r_2}(\theta_2) &\\ = \theta_2(-1)\,\Gamma_{-k}(1)s_{k+r_2}(1)\,\Gamma_{-r_2}(\theta_2^{-1})&\\ + \theta_2(-1)\sum_{\theta\neq 1}\Gamma_{-k}(\theta)s_{k+r_2}(\theta)\,\Gamma_{-r_2}\left(\frac{\theta}{\theta_2}\right)&.\end{aligned}$$

Substituting here the value of $s_{2r_2}(\theta_2)$ found above, we obtain

$$s_{k+r_2}(1)\,\Gamma_{-r_2}(\theta_2^{-1})(1 - |\mathfrak{p}| - \Gamma_{-k}(1)) = \sum_{\theta\neq 1}\Gamma_{-k}(\theta)s_{k+r_2}(\theta)\,\Gamma_{-r_2}\left(\frac{\theta}{\theta_2}\right).$$

Since $k > 0$, the coefficient of $s_{k+r_2}(1)$ on the left-hand side is different from zero. We write the conditions under which at least one term on the right-hand side may be different from zero:

$$k = \text{rank } \theta; \qquad k + r_2 = 2(\text{rank } \theta); \qquad r_2 = \text{rank}\left(\frac{\theta}{\theta_2}\right).$$

Hence, it follows that $k = \text{rank } \theta = r_2 = \text{rank}\left(\frac{\theta}{\theta_2}\right)$, and that $s_{k+r_2}(1)$ can be different from zero only for $k = r_2$. Consequently, $s_n(1) = 0$ for $n > 2$, because every $n > 2$ can be represented in the form $n = k + r_2$, $k \neq r_2$. For $n = 2$ the expression on the right may be summed explicitly, and we find

$$s_2(1) = \sum_{\theta\neq 1}\theta(-1)\,\frac{\Gamma_{-1}(\theta)\,\Gamma_{-1}\left(\frac{\theta}{\theta_2}\right)}{\Gamma_{-1}(\theta_2^{-1})} = |\mathfrak{p}|.$$

Note that all the coefficients we have found so far are uniquely determined and do not depend on the representation in question.

To determine the remaining coefficients $s_k(\theta)$, $k < 2$, we set $\theta_1 = \theta_2 = 1$ in the fundamental equation (2) and assume that at least one of the numbers k or l is nonpositive. Then we obtain the equation

But by virtue of the condition $|s(\lambda, \theta)| = 1$ the coefficients $s_k(\theta)$ satisfy the relations: $\sum_r s_{k+r}(\theta)\bar{s}_{l+r}(\theta) = \delta_{kl}$. Hence,

$$\frac{|\mathfrak{p}|}{-1+|\mathfrak{p}|} s_{k+1}(1)\bar{s}_{l-1}(1) - \sum_{r>0} s_{k+r}(1)\bar{s}_{l+r}(1) + \delta_{kl} = \frac{\Gamma_{-k}(1)\,\bar{\Gamma}_{-l}(1)}{1-|\mathfrak{p}|} s_{k-l}(1).$$

Here we set $l = 0$ and take into account that $s_r(1) = 0$ for $r > 2$:

$$\frac{|\mathfrak{p}|}{-1+|\mathfrak{p}|} s_{k+1}(1)\bar{s}_1(1) - s_{k+1}(1)\bar{s}_1(1) - s_{k+2}(1)\bar{s}_2(1) + \delta_{k0} = \frac{\Gamma_{-k}(1)}{1-|\mathfrak{p}|} s_k(1).$$

In particular, for $k = 0$ we find

$$s_0(1) + \frac{s_1^2(1)}{1-|\mathfrak{p}|} = 1 - |\mathfrak{p}|^2,$$

and for $k < 0$:

$$s_k(1) + s_{k+1}(1)\cdot\frac{\bar{s}_1(1)}{1-|\mathfrak{p}|} + s_{k-2}(1)\,|\mathfrak{p}| = 0.$$

So we have obtained a recurrence relation for $s_k(1)$. It has the general solution

$$s_k(1) = A\tau_1^k + B\tau_2^k \qquad \text{for } k \leq 1,$$

where τ_1 and τ_2 are two complex numbers linked by the condition $\tau_1\tau_2 = |\mathfrak{p}|^{-1}$.

Hence,

$$\begin{aligned} s(\lambda, 1) &= \sum_k s_k(1)\lambda^k = |\mathfrak{p}|\,\lambda^2 + \sum_{k\leqslant 1}(A\tau_1^k\lambda^k + B\tau_2^k\lambda^k) \\ &= |\mathfrak{p}|\,\lambda^2 + \frac{A\lambda\tau_1}{1-\lambda^{-1}\tau_1^{-1}} + \frac{B\lambda\tau_2}{1-\lambda^{-1}\tau_2^{-1}} \\ &= \frac{(1-\lambda\alpha)(1-\lambda\beta)}{(1-\lambda^{-1}\tau_1^{-1})(1-\lambda^{-1}\tau_2^{-1})}, \end{aligned}$$

where α and β are two complex numbers depending on A, B, τ_1, and τ_2. The condition $|s(\lambda, 1)| = 1$ is satisfied only when $\alpha = \bar{\tau}_1^{-1}$, $\beta = \bar{\tau}_2^{-1}$ or $\alpha = \bar{\tau}_2^{-1}$, $\beta = \bar{\tau}_1^{-1}$. In this case we find

$$s(\lambda, 1) = \frac{(1-\lambda\bar{\tau}_1^{-1})(1-\lambda\bar{\tau}_2^{-1})}{(1-\lambda^{-1}\tau_1^{-1})(1-\lambda^{-1}\tau_2^{-1})}.$$

Let $|\tau_1| = |\mathfrak{p}|^\rho$; by π_0 we denote the character with the coordinates $\left(\frac{\tau_1}{|\tau_1|}, 1\right)$. Then our function $s(\pi)$ may be written in the form

$$s(\pi) = \Gamma(\pi^{-1}\pi_0\pi_{(\rho)})\,\Gamma(\pi^{-1}\pi_0\pi_{(1-\rho)}).$$

We see that a representation corresponding to this function belongs to the supplementary or the singular series.

3. There are no exceptional characters.

We show that then the matrix elements of the representation are square-summable on G. In the representation space of T (in the π-realization) we choose a basis consisting of the function $e_{k,\theta}$:

$$e_{k,\theta}(\lambda, \theta_1) = \begin{cases} \lambda^k & \text{for } \theta_1 = \theta, \\ 0 & \text{for } \theta_1 \neq \theta. \end{cases}$$

In this basis the representation operators have the form

$$\begin{aligned} T(g_{a,0})e_{k,\theta} &= \theta(\alpha)e_{k-n,\theta} \qquad \text{for } a = \alpha\,|\mathfrak{p}|^n,\ \alpha \in O^*; \\ T(g_{1,b})e_{k,\theta} &= (1-|\mathfrak{p}|)^{-1}\sum_{\theta'} \Gamma_{m+k}\left(\frac{\theta}{\theta'}\right)\frac{\theta}{\theta'}(\beta)e_{k,\theta'} \qquad (11) \\ &\qquad \text{for } b = \beta\,|\mathfrak{p}|^n,\ \beta \in O^*. \\ T(s)e_{k,\theta} &= \sum_i s_t(\theta^*)\,\lambda_0^k e_{t-k,\theta^*} \end{aligned}$$

Almost every element of G can be written in the form

$$g = g_{a,0}^{-1} g_{1,b_1}^{-1} s g_{1,b_2} g_{a,0}.$$

The invariant measure on G with parameters a, b_1, b_2 is $d\mu(g) = d^*a\, db_1\, db_2$.

We examine the domain $D(n, m_1, m_2)$ on G given by the conditions

$$a = \mathfrak{p}^n\alpha,\ b_1 = \mathfrak{p}^{m_1}\beta_1,\ b_2 = \mathfrak{p}^{m_2}\beta_2 \qquad \text{where } \alpha,\ \beta_1,\ \beta_2 \in O^*.$$

Using (11) we can write down the following expression for the matrix elements of $T(g)$, with $g \in D(n, m_1, m_2)$;

$$\begin{aligned} (T(g)e_{k_1,\theta_1}, e_{k_2,\theta_2}) &= \varphi(a, b_1, b_2) \\ &= (1-|\mathfrak{p}|)^{-2}\sum_{\theta} \theta_0(\beta_2)\theta_2(\alpha\beta_2^{-1})\theta_1(-\alpha^{-1}\beta_1)\,\lambda_0^{k_2-n} \\ &\qquad \Gamma_{m_1+k_1-n}\left(\frac{\theta}{\theta_1}\right)\Gamma_{m_2+k_2-n}\left(\frac{\theta}{\theta_2^*}\right)\theta^{-1}(-\beta_1\beta_2)s_{k_1+k_2-2n}(\theta). \qquad (12) \end{aligned}$$

We investigate under what conditions on n, m_1, m_2 this expression may be different from zero.

First let n be fixed. There are only finitely many characters θ for which $s_{k_1+k_2-2n}(\theta) \neq 0$.

Indeed, the arguments we have already used in the first and second case show that for all θ, except possibly finitely many,

$$s(\lambda, \theta) = s_{\rho(\theta)}(\theta)\cdot\lambda^{\rho(\theta)}, \qquad \text{where } \rho(\theta) = 2(\text{rank } \theta) \qquad (13)$$

Our assertion now follows from the fact that the number of characters of a given rank is finite.

Next, since the rank of θ is bounded and the ranks of θ_1 and

θ_2 are fixed, for negative m_1 or m_2 of sufficiently large absolute value the coefficients of the Gamma-function on the right-hand side of (12) are zero. This means that for every fixed n the domain of the group on which our matrix element is different from zero has finite volume. For all sufficiently large positive n we may use (13) to obtain a more accurate estimate, which shows that this volume is bounded by a constant independent of n.

So far we have nowhere used the fact that there are no exceptional characters.

Now let us take this into account. First of all, it implies (by the definition of exceptional characters) that $s_k(\theta) = 0$ for $k \leq 0$.

This means that for $n \leq \dfrac{k_1 + k_2}{2}$ our matrix element φ vanishes.

In investigating the summability of φ we need therefore consider only the domain $n > N$, where N is a sufficiently large positive number. Here we may use (13) and assume that the rank of θ is greater than that of θ_1 and of θ_2^*. Hence it follows that the right-hand side of (12) is different from zero only for $m_1 = \dfrac{k_2 - k_1}{2}$, $m_2 = \dfrac{k_1 - k_2}{2}$ and has the form

$$\varphi_n(\alpha, \beta_1, \beta_2) = (1 - |\mathfrak{p}|)^{-2}\theta_0(\beta_1)\theta_1(-\alpha^{-1}\beta_1)\theta_2(\alpha\beta_2)\lambda_0^{k_2-n} \sum_{\theta} \Gamma_{-r}\left(\frac{\theta}{\theta_1}\right)\Gamma_{-r}\left(\frac{\theta}{\theta_2^*}\right)\theta(-\beta_1^{-1}\beta_2^{-1})s_{2r}(\theta),$$

where the sum is taken over all the characters of rank

$$r = \frac{k_1 + k_2}{2} + n.$$

Then,

$$|\varphi_n(\alpha, \beta_1, \beta_2)|^2 = (1 - |\mathfrak{p}|)^{-4} \sum_{\theta,\theta'} \Gamma_{-r}\left(\frac{\theta}{\theta_1}\right)\Gamma_{-r}\left(\frac{\theta}{\theta_2^*}\right)s_{2r}(\theta)\,\overline{\Gamma_{-r}\left(\frac{\theta'}{\theta_1}\right)\Gamma_{-r}\left(\frac{\theta'}{\theta_2^*}\right)s_{2r}(\theta')}\,{}^{-1}\theta\theta'(-\beta_1\beta_2).$$

Therefore, the integral of $|\varphi(g)|^2$ over the domain $\bigcup\limits_{m_1, m_2} D(n, m_1, m_2)$ is equal to

$$\int |\varphi_n(\alpha, \beta_1, \beta_2)|^2\, d\beta_1\, d\beta_2 = (1 - |\mathfrak{p}|)^{-4} \sum_{\theta} \left| \Gamma_{-r}\left(\frac{\theta}{\theta_1}\right)\Gamma_{-r}\left(\frac{\theta}{\theta_2^*}\right)s_{2r}(\theta)\right|^2.$$

Bearing in mind that $|s_{2r}(\theta)| = 1$, $|\Gamma_{-r}(\theta)| = |\mathfrak{p}|^{r/2}$ and that the number of characters of rank r is $|\mathfrak{p}|^{-r}(1 - |\mathfrak{p}|)$ we obtain the required estimate.

This concludes the proof of Theorem 4′.

We mention that by other straightforward arguments it can be shown that $\varphi(g) \in L^p(G)$ for every $p \geq 1$.

REPRESENTATIONS OF ADELE GROUPS

3

§ 1. ADELES AND IDELES

1. The Group of Characters of the Additive Group of Rational Numbers. To get a better understanding of the structure of the character group of the additive group of rational numbers, we consider first a significantly simpler problem. Namely, we clarify the structure of the character group of the group $Q^{(p)}$ of all fractions of the form $\frac{a}{p^n}$, where p is a fixed prime number and a and n integers.

Let $\chi\left(\frac{a}{p^n}\right)$ be an arbitrary character on $Q^{(p)}$. Since

$$\chi\left(\frac{a}{p^n}\right) = \left[\chi\left(\frac{1}{p^n}\right)\right]^a, \tag{1}$$

to determine χ it is sufficient to know the numbers $\chi(1)$, $\chi\left(\frac{1}{p}\right), \ldots,$ $\chi\left(\frac{1}{p^n}\right), \ldots$. These numbers are connected by the following relations:

$$\left[\chi\left(\frac{1}{p^{n+1}}\right)\right]^p = \chi\left(\frac{1}{p^n}\right), \qquad n = 0, 1, \ldots \tag{2}$$

Conversely, every choice of numbers $\chi\left(\frac{1}{p^n}\right)$, $n = 0, 1, \ldots,$ satisfying the relations (2) and such that $\left|\chi\left(\frac{1}{p^n}\right)\right| = 1$, yields a character χ on $Q^{(p)}$ which is determined by formula (1).

Since $\left|\chi\left(\frac{1}{p^n}\right)\right| = 1$, we have

$$\chi\left(\frac{1}{p^n}\right) = \exp 2\pi i \frac{\alpha_n}{p^n}, \tag{3}$$

where α_n is a real number determined mod p^n. The relations (2) are equivalent to the following relations for the numbers α_n:

$$\alpha_{n+1} \equiv \alpha_n (\operatorname{mod} p^n).$$

Hence, it follows that

$$\alpha_n = -\alpha + \beta_n,$$

where $\alpha = -\alpha_0$ and the β_n are integers determined mod p^n and satisfying the relation

$$\beta_{n+1} \equiv \beta_n (\operatorname{mod} p^n). \tag{4}$$

From (4) it follows that the β_n are segments of the p-adic series

$$\beta = a_1 + a_2 p + a_3 p^2 + \dots 0 \leq a_k < p.$$

Thus,

$$\chi\left(\frac{1}{p^n}\right) = \exp 2\pi i \frac{-\alpha + \beta}{p^n}.$$

But then we find from (1) for every rational fraction of the form $\frac{a}{p^n}$ that

$$\chi\left(\frac{a}{p^n}\right) = \exp 2\pi i(-\alpha + \beta) \frac{a}{p^n}. \tag{5}$$

So every character $\chi\left(\frac{a}{p^n}\right)$ of the group $Q^{(p)}$ is given by a pair of numbers—a real number α mod 1 and a p-adic integer β.

It is not hard to check that $\chi\left(\frac{a}{p^n}\right) \equiv 1$ if and only if α is a rational number whose denominator is not divisible by p, and $\beta = \alpha$ (β is the p-adic representation of the rational number α).

Thus, the character group of the additive group of all fractions of the form $\frac{a}{p^n}$ has the following structure. We take the additive group whose elements are the pairs (α, β), where α is a real number mod 1 and β a p-adic integer. In this group we factor out the subgroup of elements of the form (r, r), where r ranges over all rational numbers whose denominators are not divisible by p.

The structure of the character group of the additive group Q of all rational numbers turns out to be essentially more complicated. This character group will be investigated in detail in the next few subsections. Here we only state the final result.

Every character of Q is given by an infinite sequence

$$a = (a_\infty, a_2, \dots, a_p, \dots). \tag{6}$$

in which a_∞ is a real number, and a_p a p-adic number ($p = 2, 3, \dots$), and all the a_p beginning with a sufficiently large p are p-adic integers. Such sequences are called *adeles*.

The character $\chi_a(r)$ corresponding to the adele a is given by the following formula:

$$\chi_a(r) = \exp 2\pi i(-a_\infty r + a_2 r + \dots + a_p r + \dots). \tag{7}$$

This formula must be understood in the following sense. Since an arbitrary integer can be added to the expression in parentheses, we ignore the integral parts of all the p-adic numbers $a_p r$ on the right. Then the sum in parentheses becomes the sum of rational numbers. It is easy to check that here only finitely many terms are different from zero.

The set of adeles forms an additive group A if the operation of addition is defined componentwise. Clearly, when two adeles are added, their corresponding characters are multiplied. Thus, the map

$$a \to \chi_a(r)$$

is a homomorphism of the group of adeles A onto the group Q' of characters of Q.

Let us find the kernel of this homomorphism. It turns out that $\chi_a(r) \equiv 1$ if and only if a has the following form:

$$a = (\alpha, \alpha, \dots, \alpha, \dots),$$

where α ranges over the rational numbers. Such sequences a are called *principal* adeles. Obviously, the subgroup of principal adeles is isomorphic to the additive group Q of rational numbers; we denote it also by the letter Q.

Then we have an isomorphism

$$Q' \cong A/Q$$

between the character group Q' and the factor group A/Q of the group of adeles with respect to the subgroup of principal adeles. All the groups so far are regarded as abstract. It can be shown, however, that under a natural topology in the group of adeles the isomorphism is one of topological groups. The detailed proof of all the statements we have made here will be given in **§ 1.6**.

2. Definition of Adeles and Ideles. We repeat the definition of adeles. We consider the collection A of all sequences of the form

$$a = (a_\infty, a_2, \dots, a_p, \dots),$$

where a_∞ is a real number, the a_p are p-adic numbers and all the a_p beginning from a certain p onward (depending on a) are p-adic integers. The set of all such sequences forms a ring under componentwise addition and multiplication. This is called the *ring of adeles*, and its additive group the *group of adeles*.

The elements of the ring of adeles A that have an inverse are called *ideles*. The set A^* of all ideles forms a group under multiplication. It is called the group of *ideles*.

Thus, the elements of the group of ideles are sequences

$$\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots),$$

where $\lambda_p \neq 0$ and $|\lambda_p|_p = 1$ for all p with a finite number of exceptions ($|x|_p$ is the p-adic norm).

We introduce a topology in the group of adeles A in the following way. We take the subgroup A° of the adeles

$$a = (a_\infty, a_2, \ldots, a_p, \ldots),$$

where all the a_p are p-adic integers. In A° we introduce the topology of the Tikhonov product of the topological spaces $R, O_2, \ldots, O_p, \ldots$, where O_p is the subgroup of p-adic integers. This subgroup A° is declared to be an open set in A.

Thus, a sequence of adeles $a^{(n)} = (a_\infty^{(n)}, a_2^{(n)}, \ldots, a_p^{(n)}, \ldots)$ is said to converge to the adele $a = (a_\infty, a_2, \ldots, a_p, \ldots)$ if it converges to a componentwise and if there is an N such that for $n \geq N$ the numbers $a_p - a_p^{(n)}$ are p-adic integers.

The topological group A so obtained is locally compact; this follows immediately from the compactness of the groups O_p.

Similarly we introduce a topology in the group of ideles A^*.

3. Another Construction of the Group of Adeles. Let Q be the additive group of rational numbers. We introduce in Q a topology by taking as neighborhoods of zero all the subgroups of Q. We show that *the completion $\bar{Q}$ of Q relative to this topology is isomorphic to the group of all adeles of the form*

$$(0, a_2, \ldots, a_p, \ldots).$$

Thus, the group of adeles A is the direct product

$$A = A_\infty \times \bar{Q}$$

of the group of real numbers A_∞ and of $\bar{Q}$.

Proof. We consider the subgroup B of adeles of the form $(0, a_2, \ldots, a_p, \ldots)$. With every rational number r we associate the sequence $(0, r, \ldots, r, \ldots)$. This sequence is an element of B, because $|r|_p = 1$ for sufficiently large p (namely, for those p that do

not occur as factors of r). Thus, the correspondence

$$r \to (0, r, \ldots, r, \ldots)$$

is an isomorphic embedding of Q in B.

Let us show that this embedding induces in Q a topology that coincides with the original one.

For this purpose we examine in B the open subgroups $U_{p,n}$ consisting of the adeles $(0, a_2, \ldots, a_p, \ldots)$ for which $|a_q|_q \leq q^{-n}$ for $q \leq p$ and $|a_q|_q \leq 1$ for $q > p$. Clearly, the $U_{p,n}$ form in B a complete system of neighborhoods of zero. The intersection $U_{p,n} \cap Q$ consists of the integers of the form $(2 \cdot 3 \cdot 5 \cdots p)^n k$, where k ranges over all integers: in other words, $U_{p,n} \cap Q$ is a cyclic subgroup of Q. These subgroups are open sets in the original topology of Q. Moreover, they form a complete system of neighborhoods of zero in Q, because every subgroup of Q contains a subgroup of this form. So we see that the topology of B induces the original topology in $Q \subset B$.

We have to show that the set of elements of Q is everywhere dense in B, that is, that for every element

$$a = (0, a_2, \ldots, a_p, \ldots)$$

of B there exists a sequence of elements in Q converging to a.

For every prime number p and every natural number n we denote by $b_{q,p,n}$ a fractional part of the q-adic number $(p!)^{-n}a_q$. We set

$$c_{p,n} = \sum_q b_{q,p,n}.$$

Obviously in this sum only finite by many terms are different from zero. Then we have for every prime number q

$$|(p!)^{-n}a_q - c_{p,n}|_q \leq 1$$

Hence, it follows that

$$\begin{aligned} |a_q - (p!)^n c_{p,n}|_q &\leq q^{-n} \qquad \text{for} \qquad q \leq p, \\ |a_q - (p!)^n c_{p,n}| &\leq 1 \qquad \text{for} \qquad q > p. \end{aligned}$$

This means that the sequence of rational numbers $(p!)^n c_{p,n}$ converges to the adele a, as $p \to \infty$, $n \to \infty$.

4. The Isomorphisms $Q \to A$ and $Q^* \to A^*$. We show that the ring of rational numbers Q can be isomorphically embedded in the ring of adeles A.

For Q is isomorphically embedded in the field of real numbers R as well as in the field of p-adic numbers Q_p. With every rational number r we associate the sequence

$$(r, r, \ldots, r, \ldots).$$

These sequences are adeles, because for $r \neq 0$ we have $|r|_p = 1$ for sufficiently large p. They are called principal adeles.

Let us show that *the ring Q of principal adeles is discrete in A.*

Proof. We assume the contrary: that Q is not discrete in A. Then we can find a sequence of principal adeles $r_n = (r_n, r_n, \ldots, r_n, \ldots)$ converging to zero. From the definition of the topology in A it now follows that beginning with a sufficiently large n the number r_n is a p-adic integer for every p. But this means that r_n is an integer and therefore the sequence r_n does not converge to zero in the topology of R. So we have arrived at a contradiction.

Let us find a fundamental domain of the additive group of A relative to the subgroup of principal adeles Q.

Consider the set F of adeles

$$a = (a_\infty, a_2, \ldots, a_p, \ldots).$$

where $0 \leq a_\infty < 1$ and $|a_p|_p \leq 1$; we show that this is the required fundamental domain.

Let $a = (a_\infty, a_2, \ldots, a_p, \ldots)$ be an arbitrary adele. We denote by α the sum of the fractional parts of the p-adic numbers $a_2, \ldots, a_p, \ldots$. By the definition of an adele, this sum contains only finitely many terms different from zero, hence is a rational number. Now we choose an integer n such that $0 \leq a_\infty - \alpha - n < 1$ and we consider the principal adele

$$r = (r, r, \ldots, r, \ldots),$$

where $r = \alpha + n$. Obviously, $a - r \in F$.

So we have shown that A is a union of sets $r + F$, where r ranges over the principal adeles. We show that these sets are pairwise disjoint.

Let $r \neq 0$; then the sets $r + F$ and F have no common elements. For if r is an integer, then $r + a_\infty$, where $a \in F$, does not belong to the half-open interval $0 \leq x < 1$; hence $r + a$ does not belong to F. But if r is not an integer, then $|r|_p > 1$ for at least one p, and therefore $|r + a_p|_p > 1$, where $a \in F$; consequently, in this case $r + a$ also claims to belong to F.

So we have shown that F is a fundamental domain.

Note that this is a compact set. From this it follows that *the factor group A/Q of a group of adeles by the subgroup of principal adeles is compact.* In **§ 1.6** we shall show that this factor group is isomorphic to the character group of Q.

Now we pass on to the group of ideles. The multiplicative group of rational numbers Q^* can be isomorphically embedded in the group of ideles A^*. Namely, with every rational number $r \neq 0$ we associate the sequence

These sequences are ideles, because $|r|_p = 1$ for all p that do not occur in the prime decomposition of r. They are called *principal ideles*.

We note that the set Q^* of principal ideles is obtained from the set Q of principal adeles by omitting the zero adele $(0, 0, \ldots, 0, \ldots)$.

As in the case of adeles, it is easy to check that the subgroup Q^* of principal ideles is discrete in the group A^* of all ideles.

5. The Group of Additive Characters of the Ring of Adeles A. We begin by explaining the important concept of a self-dual ring.

Let L be the commutative topological ring with a unit element. Consider the set L' of all additive characters of L, that is, of those continuous functions $\chi(x)$ on L for which $|\chi(x)| \equiv 1$, and

$$\chi(x + y) = \chi(x)\chi(y)$$

for any x and y in L.

There is a natural way of introducing in L' the structure of a topological space and an operation of multiplication of characters under which L' becomes a topological group. Furthermore, there is in L' a natural definition of the operation of multiplication by elements from the original ring L: the product $a \cdot \chi$ of the element $a \in L$ and the character $\chi(x)$ is the character $\chi_a(x) \equiv \chi(ax)$.

Clearly, this operation of multiplication is continuous in a and χ and satisfies the following conditions

$$a \cdot (\chi_1\chi_2) = (a \cdot \chi_1)(a \cdot \chi_2),$$

$$(a + b)\chi = (a \cdot \chi)(b \cdot \chi),$$

$$(ab)\chi = a \cdot (b \cdot \chi),$$

$$1 \cdot \chi = \chi.$$

Thus, the set of characters L' is an L-module.

The ring L is called self-dual if the L-module of all additive characters on L is isomorphic to the original ring L.

An equivalent definition is: *the ring L is called self-dual if every additive character $\chi(x)$ is of the form*

$$\chi(x) = \chi_0(ax), \qquad a \in L.$$

where χ_0 is a fixed character.

Examples of self-dual rings are well known to us: all topological locally compact fields are of this kind. On the other hand, infinite discrete fields, particularly the field Q of rational numbers, are not self-dual (because the set of their additive characters is not compact).

We now show that *the ring of adeles A is self-dual.*

To prove this we introduce an additive character $\chi_0(x)$ on A. It plays a fundamental role in what follows.

Consider the function $\sigma(a)$ on A, with values in the group of real numbers mod 1, defined by the following formula:

$$\sigma(a) \equiv -a_\infty + a_2 + \ldots + a_p + \ldots \pmod 1. \tag{1}$$

In other words, $\sigma(a) + a_\infty$ denotes the sum of the fractional parts of the p-adic numbers a_p (note that the fractional part of a p-adic number a_p is an ordinary rational number). Since all the numbers a_p beginning with sufficiently large p are p-adic integers, only finitely many terms in this sum are different from zero. Hence, the sum always has a meaning.

We establish some properties of this function $\sigma(a)$. First, from the definition it follows immediately that

$$\sigma(a + a') = \sigma(a) + \sigma(a') \tag{2}$$

for arbitrary $a, a' \in A$.

Next, it is obvious that on every subring Q_p of A (that is, on the subring of adeles of the form $(0, \ldots, 0, a_p, 0, \ldots)$) we have $\sigma(a_p) = 0$ if and only if a_p is a p-adic integer.

Finally, we show that *if a is a principal adele, then* $\sigma(a) = 0$.

For let a be a principal adele, that is, $a = (r, r, \ldots, r, \ldots)$, where r is a rational number. This number can always be represented in the form of a sum

$$r = \frac{\alpha_2}{2^{n_2}} + \frac{\alpha_3}{3^{n_3}} + \ldots + \frac{\alpha_p}{p^{n_p}} + \ldots$$

where the α_p are integers; the number of nonzero terms in this sum is finite. Thus, the fractional part of r regarded as a p-adic number is $\dfrac{a_p}{p^{n_p}}$ mod 1, and so

$$\sigma(a) \equiv -r + \frac{\alpha_2}{2^{n_2}} + \ldots + \frac{\alpha_p}{p^{n_p}} + \ldots \pmod 1$$

that is, $\sigma(a) = 0$.

We define a function $\chi_0(a)$ on A by the following formula:

$$\chi_0(a) = \exp 2\pi i\sigma(a) \equiv \exp 2\pi i(-a_\infty + a_2 + \ldots + a_p + \ldots). \tag{3}$$

By what we have proved above, $\chi_0(a)$ is an additive character on A and satisfies the following conditions:

1. $\chi_0(a_p) = 1$ on every subring Q_p of A if and only if a_p is a p-adic integer.

2. $\chi_0(a) \equiv 1$ on the subring Q of principal adeles.

We now proceed to the proof of the assertion that A is a self-dual ring. We shall show that every additive character $\chi(a)$ on A has the form

$$\chi(a) = \chi_0(ba),$$

where $b \in A$.

As a preliminary we observe that every character $\chi(a)$ on A can be written as a convergent infinite product

$$\chi(a) = \chi(a_\infty)\chi(a_2) \ldots \chi(a_p), \ldots \tag{4}$$

where $\chi(a_p)$ is the restriction of χ to Q_p.

For we have $a = \lim_{p\to\infty} a^{(p)}$, where

$$a^{(p)} = (a_\infty, a_2, \ldots, a_p, 0, 0, \ldots).$$

Consequently, since $\chi(a)$ is a continuous function,

$$\chi(a) = \lim_{p\to\infty} \chi(a^{(p)}) = \lim_{p\to\infty} [\chi(a_\infty)\chi(a_2) \cdots \chi(a_p)].$$

Now let $\chi(a)$ be an arbitrary character on A, which we represent in the form (4). Since $\chi_0(a_p) \equiv 1$ on Q_p, by what was proved in Chapter II, the character $\chi(a_p)$ on Q_p can be represented in the form

$$\chi(a_p) = \chi_0(b_pa_p), \tag{5}$$

where $b_p \in Q_p$. Here the element b_p is uniquely determined by χ_0. So we have

$$\chi(a) = \chi_0(b_\infty a_\infty)\chi_0(b_2a_2) \cdots \chi_0(b_pa_p) \tag{6}$$

Now we show that

$$b = (b_\infty, b_2, \ldots, b_p, \ldots)$$

is an adele; this also proves that $\chi(a) = \chi_0(ba)$. Suppose that b is not an adele. This means that there are infinitely many numbers $b_{p_1}, b_{p_2}, \ldots, b_{p_k}, \ldots$ that are not p-adic integers. By the property of χ_0 we have

$$\chi_0(b_{p_k}a_{p_k}) \equiv 1,$$

where a_{p_k} ranges over the p_k-adic integers. Therefore, we can choose a p_k-adic integer a_{p_k} such that

$$|\chi_0(b_{p_k}a_{p_k}) - 1| > \varepsilon,$$

where ε is a fixed number, the same for all p_k (for example, we can take $\varepsilon = 1/2$).

Clearly, the infinite product

$$\prod_{k=1}^{\infty} \chi_0(b_{p_k}a_{p_k})$$

then does not converge; this contradicts (6) and proves the assertion.

6. The Characters of the Group A/Q. We now show that *the character group of the factor group A/Q, where Q is the subgroup of principal adeles, is isomorphic to the additive group of rational numbers.*

By the Pontryagin duality theorem, it follows that, conversely, the character group of the additive group of rational numbers is isomorphic to A/Q. This important fact was already stated without proof in **§ 1.1.**

To prove it we observe that the character group of A/Q is isomorphic to the subgroup of all characters $\chi(a)$ on A for which

$$\chi(a) \equiv 1 \quad \text{for every } a \in Q.$$

According to a result in **§ 1.5**, every character $\chi(a)$ has the form

$$\chi(a) = \chi_0(ba),$$

where $b \in A$ and

$$\chi_0(a) = \exp 2\pi i(-a_\infty + a_2 + \dots + a_p + \dots).$$

Hence, we seek those $b \in A$ for which the condition

$$\chi_0(ba) \equiv 1 \quad \text{for every } a \in Q \tag{1}$$

is satisfied.

From **§ 1.5** we already know that $\chi_0(a) \equiv 1$ on Q. Thus, if $b \in Q$, then the character $\chi_0(ba)$ satisfies condition (1). It is not difficult to verify that the converse also holds, namely, if $\chi_0(ba)$ satisfies condition (1), then $b \in Q$.

So we have shown that the character group of A/Q is isomorphic to the group of rational numbers.

7. Invariant Measures in the Group of Adeles and the Group of Ideles. The group of adeles A, being locally compact, has an invariant measure which we denote by da. This measure will always be normalized by the following condition:

$$\int_F da = 1, \tag{1}$$

where the integral is taken over the compact set F of adeles

$$a = (a_\infty, a_2, \dots, a_p, \dots)$$

for which

$$0 \le a_\infty \le 1, \quad |a_p|_p \le 1, \quad p = 2, 3, \dots.$$

It is easy to check that the measure da can be expressed in terms of the measures da_p on the groups Q_p as follows:

$$da = da_\infty da_2 \dots da_p \dots, \tag{2}$$

where the measures da_p are normalized by the conditions

$$\int_0^1 da_\infty = 1, \qquad \int_{|a_p|_p \leq 1} da_p = 1.$$

The equation (2) must be understood in the following sense: if $\varphi(a)$ is a summable function of A of the form

$$\varphi(a) = \varphi_\infty(a_\infty)\varphi_2(a_2) \cdots \varphi_p(a_p) \cdots,$$

then

$$\int \varphi(a)\, da = \int \varphi_\infty(a_\infty)\, da_\infty \int \varphi_2(a_2)\, da_2 \ldots \int \varphi_p(a_p)\, da_p \ldots .$$

Similarly, in the group of ideles A^* there exists an invariant measure which we denote by $d^*\lambda$. We assume that this measure is normalized by the following condition:

$$\int_{\Lambda'} d^*\lambda = 1, \tag{3}$$

where Λ' is the compact set of all ideles of the form

$$\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots),$$

$1 \leq \lambda_\infty \leq e$ (e is the base of the natural logarithms), $|\lambda_p|_p = 1$, $p = 2, 3, \ldots$.

It is easy to verify that the measure $d^*\lambda$ can be expressed in terms of the measures $d^*\lambda_p$ on the multiplicative group Q_p^* as follows:

$$d^*\lambda = d^*\lambda_\infty\, d^*\lambda_2 \cdots d^*\lambda_p \cdots \tag{4}$$

where $d^*\lambda_\infty = \dfrac{d\lambda_\infty}{|\lambda_\infty|_\infty}$, $d\lambda_\infty$ being the usual measure on the real line and the measures $d^*\lambda_p$ being normalized by the condition†

$$\int_{|\lambda_p|_p = 1} d^*\lambda_p = 1.$$

8. The Function $|\lambda|$. It is easy to see that every idele λ effects an isomorphic map

$$a \to \lambda a$$

† Note that here the normalization is different from the one taken in Chapter II. There we set $d^*\lambda_p = \dfrac{d\lambda_p}{|\lambda_p|_p}$, where $d\lambda_p$ is the measure of the additive group of the field Q_p, normalized by the condition $\int_{|\lambda_p| \leq 1} d\lambda_p = 1$. Here we set

$$d^*\lambda_p = (1 - p^{-1})^{-1} \frac{d\lambda_p}{|\lambda_p|_p}.$$

of the group of adeles A onto itself. Therefore, if da is an invariant measure on A, then $d_\lambda(a) = d(\lambda a)$ is also an invariant measure on A, and therefore proportional to da. We denote the factor of proportionality by $|\lambda|$.

Thus, *the function* $|\lambda|$ *on the group of ideles* A^* *is defined by the following formula:*

$$d(\lambda a) = |\lambda|\, da, \tag{1}$$

where da is an invariant measure on the group of adeles A.

From the definition it follows immediately that

$$|\lambda' \lambda''| = |\lambda'| \cdot |\lambda''| \tag{2}$$

for arbitrary λ', $\lambda'' \in A^*$.

Let us find an explicit expression for $|\lambda|$. For this purpose we use formula (2) in **§ 1.7**, which expresses da in terms of the measures da_p

$$da = da_\infty\, da_2 \ldots da_p \ldots \tag{3}$$

Hence, we have

$$d(\lambda a) = d(\lambda_\infty a_\infty)\, d(\lambda_2 a_2) \ldots d(\lambda_p a_p) \ldots \tag{4}$$

but

$$d(\lambda_p a_p) = |\lambda_p|_p\, da_p .$$

Consequently, equating (3) and (4) we find

$$|\lambda| = |\lambda_\infty|\, \lambda_2|_2 \ldots |\lambda_p|_p \ldots \tag{5}$$

Note that in this infinite product all but a finite number of factors are equal to 1.

We shall now establish the following important property of $|\lambda|$: *if λ is a principal idele, then*

$$|\lambda| = 1.$$

For let λ be a principal idele, that is,

$$\lambda = (r, r, \ldots, r, \ldots),$$

where r is a rational number. We decompose r into prime factors:

$$r = 2^{n_2} 3^{n_3} \ldots p^{n_p} \ldots$$

where the n_p are integers and all n_p but a finite number are zero. Then we have $|r|_p = p^{-n_p}$. Therefore, $\prod_p |r_p|_p = |r|_\infty^{-1}$ and so $|\lambda| = 1$.

9. The Characters of the Group of Ideles A^*. We give a description of the characters of the group of ideles

$$\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots),$$

which we denote by $\pi(\lambda)$.

Let $\pi_p(\lambda_p)$ be the restriction of $\pi(\lambda)$ to the subgroup Q_p^* of ideles of the form $(1, \ldots, 1, \lambda_p, 1, \ldots)$, $p = \infty, 2, 3, \ldots$. Let us show that *the character $\pi(\lambda)$ can be expressed as a convergent product*

$$\pi(\lambda) = \pi_\infty(\lambda_\infty)\pi_2(\lambda_2) \ldots \pi_p(\lambda_p) \ldots . \tag{1}$$

For consider the sequence of ideles of the form

$$\lambda^{(p)} = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, 1, 1, \ldots).$$

By the definition of the topology of A^*, this sequence converges to the idele λ. Consequently,

$$\pi(\lambda) = \lim_{p\to\infty} \pi(\lambda^{(p)}) = \lim_{p\to\infty} [\pi_\infty(\lambda_\infty)\pi_2(\lambda_2) \cdots \pi_p(\lambda_p)].$$

Thus, by virtue of (1), the character $\pi(\lambda)$ is completely determined by the family of characters $\pi_p(\lambda_p)$ defined on the groups Q_p^*. We say that this character $\pi(\lambda)$ is the *tensor product* of the characters $\pi_p(\lambda_p)$.

Now we raise the converse problem. Let $\pi_\infty(\lambda_\infty), \pi_2(\lambda_2), \ldots, \pi_p(\lambda_p), \ldots$ be a preassigned sequence of characters. The question is under what condition on these characters formula (1) gives us the character $\pi(\lambda)$ on A^*, so that the infinite product (1) converges.

We show that *formula* (1) *determines a character on the group* A^* *if and only if the characters $\pi_p(\lambda_p)$ satisfy the following condition:*

For all but a finite number of primes p

$$\pi_p(\lambda_p) \equiv 1, \qquad \text{when} \quad |\lambda_p|_p = 1.$$

Suppose that the condition holds. We consider an arbitrary idele $\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots)$. From the definition of an idele it follows that $|\lambda_p|_p = 1$ when p is sufficiently large. But then, by this condition we have $\pi_p(\lambda_p) = 1$, when p is sufficiently large. Therefore, in (1) only finitely many factors are different from zero and so the product converges.

Conversely, let us assume that the condition does not hold. Then there exists a sequence of prime numbers $p_1, p_2, \ldots, p_k, \ldots$ for which

$$\pi_{p_k}(\lambda_{p_k}) \not\equiv 1, \qquad \text{when} \quad |\lambda_{p_k}|_{p_k} = 1.$$

Obviously, in this case we can find for every p_k a λ_{p_k} such that $|\lambda_{p_k}|_{p_k} = 1$ and $|\pi_{p_k}(\lambda_{p_k}) - 1| > 1/2$. But then the product

$$\prod_k \pi_{p_k}(\lambda_{p_k})$$

diverges.

So we have obtained the following final result.

Every character π on A^* is given by a sequence of characters

$$\pi = (\pi_\infty, \pi_2, \ldots, \pi_p, \ldots),$$

where π_∞ is a multiplicative character in the field of real numbers, π_p a multiplicative character in the field of p-adic numbers, and for sufficiently large p we have $\pi_p(\lambda_p) = 1$ when $|\lambda_p|_p = 1$ (and hence, $\pi_p(\lambda_p) = |\lambda_p|^{\nu_p}$, where ν_p is some complex number).

The value of the character π on the idele

$$\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots)$$

can be expressed in the form of the following infinite product:

$$\pi(\lambda) = \pi_\infty(\lambda_\infty)\pi_2(\lambda_2) \cdots \pi_p(\lambda_p) \cdots.$$

10. The Characters of the Group A^*/Q^*. We denote by Λ the subgroup of all ideles

$$\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots),$$

in which the real number λ_∞ is positive and $|\lambda_p|_p = 1$ for all prime numbers p.

We show that *the group of ideles A^* splits into the direct product*

$$A^* = Q^* \times \Lambda \tag{1}$$

of the subgroup of principal ideles Q^* and the subgroup Λ.

Proof. Let $\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots)$ be an arbitrary idele. We represent the number λ_p in the form

$$\lambda_p = |\lambda_p|_p^{-1}\, \lambda_p'.$$

Let

$$q = \operatorname{sign} \lambda_\infty \prod_p |\lambda_p|_p^{-1}.$$

(In this product only finitely many factors are different from 1 so that q is a rational number.) By the same symbol q we denote the corresponding principal idele:

$$q = (q, q, \ldots, q, \ldots).$$

Obviously, $\lambda q^{-1} \in \Lambda$. This shows that the group of ideles A^* is the product of the subgroups Λ and Q^*.

We show that the subgroups Λ and Q^* have no common elements except the unit element. Suppose then that the principal idele $\alpha = (\alpha, \alpha, \ldots, \alpha, \ldots)$ belongs to Λ. Then for every prime number p we have $|\alpha|_p = 1$; consequently, $\alpha = \pm 1$. But we must have $\alpha > 0$; hence, $\alpha = 1$, and the assertion is proved.

From the decomposition $A^* = Q^* \times \Lambda$ it follows that the *factor group A^*/Q^* of the group of ideles A^* by the subgroup Q^* of principal ideles is isomorphic to Λ:*

$$A^*/Q^* \cong \Lambda.$$

Hence, this factor group has a very simple structure: it is the

topological direct product of the multiplicative group of all positive real numbers and the groups O_p^* of p-adic numbers of norm 1.

Observe that in contrast to the case of the group of adeles the factor group $A^*/Q^* \cong \Lambda$ is not compact.

We now proceed to a description of the characters of the group $\Lambda \cong A^*/Q^*$. It is obvious that *a character $\pi(\lambda)$ on Λ is given when the character is given on each of its direct factors, namely, the character $\pi_\infty(\lambda_\infty)$ on the group of positive real numbers, and the characters $\theta_p(\lambda_p)$ on the groups O_p^* of p-adic numbers of norm* 1. *Here, all but a finite number of characters θ_p must be identically equal to unity.*

The value of the character $\pi(\lambda)$ on the idele

$$\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots)$$

of Λ is expressed by the following formula:

$$\pi(\lambda) = \pi_\infty(\lambda_\infty)\theta_2(\lambda_2) \ldots \theta_p(\lambda_p) \ldots \tag{2}$$

We recall that every character $\pi_\infty(\lambda_\infty)$ on the group of positive real numbers has the form

$$\pi_\infty(\lambda_\infty) = \lambda_\infty^s,$$

where s is an arbitrary complex number (here we do not assume that the characters are unitary). Thus, formula (2) can be rewritten in the following form

$$\pi(\lambda) = \lambda_\infty^s\theta(\lambda) = |\lambda|^s\theta(\lambda), \tag{3}$$

where

$$|\lambda| = |\lambda_\infty|_\infty\, |\lambda_2|_2 \ldots |\lambda_p|_p \ldots,$$

$$\theta(\lambda) = \theta_2(\lambda_2) \ldots \theta_p(\lambda_p).$$

We mention that the characters $\theta(\lambda)$ are characters on the compact group of ideles of the form $(1, \lambda_2, \ldots, \lambda_p, \ldots)$; consequently, they form a discrete (countable) set. Hence, the set of characters π on A^*/Q^* can be regarded as a countable family of planes of the complex variable s (the "suffix" of the plane is given by the character θ). This enables us to introduce the concept of an analytic function of π. We say that $f(\pi) = f(s, \theta)$ is an analytic function of π if for every fixed θ it is an analytic function of the complex variable s.

Now we describe the set of characters on the group of ideles A^* that are identically equal to unity on the subgroup of principal ideles Q^*. By the decomposition

$$A^* = Q^* \times \Lambda,$$

every such character is obtained by the following construction. We take an arbitrary character $\pi(\lambda)$ on Λ and then extend it to a

character π on the group A^* according to the following formula:

$$\pi(\lambda) = \pi\left(\frac{\lambda}{q}\right). \tag{4}$$

Here $q = \operatorname{sign} \lambda_\infty \cdot \prod_p |\lambda_p|_p^{-1}$ is the component of λ in Q^*.

On the basis of (3) and (4) we obtain the following result. *Every character $\pi(\lambda)$ on the group of ideles of A^* and identically equal to unity on the subgroup of principal ideles Q^* has the following form*

$$\pi(\lambda) = |\lambda|^s\, \theta(\lambda).$$

Here s is an arbitrary complex number,

$$|\lambda| = |\lambda_\infty|_\infty\, |\lambda_2|_2 \ldots |\lambda_p|_p \ldots, \text{ and } \theta(\lambda) = \theta_2(\lambda_2) \ldots \theta_p(\lambda_p) \ldots$$

is an arbitrary character on the subgroup of ideles of the form $(1, \lambda_2, \ldots, \lambda_p, \ldots)$, where $|\lambda_p|_p = 1$ for every p. The character $\theta(\lambda)$ is assumed to be extended to the whole group A^* by the following formula:

$$\theta(\lambda) = \theta_2(q^{-1}\lambda_2) \cdots \theta_p(q^{-1}\lambda_p) \cdots, \tag{5}$$

where

$$q = \operatorname{sign} \lambda_\infty \prod_p |\lambda_p|_p^{-1} = \frac{\lambda_\infty}{|\lambda|}. \tag{6}$$

APPENDIX TO § 1

On a Zeta-Function

For every vector $\xi = (\xi_1, \ldots, \xi_n) \in A^n$ we define a norm $|\xi|$ as follows:

$$|\xi| = \prod_p |\xi|_p, \qquad \xi = (\xi_1, \ldots, \xi_n),$$

where

$$|\xi|_p = \max_k |\xi_k|_p.$$

Obviously, $|q\xi| = |\xi|$ for every $q \in Q^*$. As we know,

$$|\xi| = 1,$$

if $n = 1$ and $\xi \in Q^*$. If $n > 1$ and $\xi \in Q^n$, then it is easy to check that $|\xi| < \infty$; however, in general, $|\xi| \neq 1$.

In this appendix we investigate a series that indicates by how much the relation $|\xi| = 1$ for $\xi \in Q^n$ fails to hold.

We denote by T the collection of vectors $\xi \in Q^n$ in which all the components are different from zero. Consider the series

$$D(s) = \sum_{T/Q^*} |t|^{-s}. \qquad (1)$$

We show that it converges; at the same time, we evaluate it.

Every vector $t \in T$ can be brought by multiplication by $q \in Q^*$ into the form

$$t = (t_1, \ldots, t_n), \qquad (2)$$

where $t_1, \ldots, t_n$ are integers whose greatest common divisor is 1.

Obviously all the vectors (2) that differ in sign are inequivalent, that is, cannot be obtained from each other by multiplication by an element $q \in Q^*$.

Thus, we have

$$D(s) = \sum_{m=1}^{\infty} \frac{F(m)}{m^s}, \qquad (3)$$

where $F(m)$ is the number of n-dimensional vectors t with coprime integer coordinates and $|t| = m$.

It is not hard to check that $F(m) \leq nm^{n-1}$. Consequently, the series for $D(s)$ converges for $\operatorname{Re} s > n$.

We denote by $F_+(m)$ the number of $(n-1)$-dimensional vectors with positive coordinates not exceeding m and such that their greatest common divisor is prime to m.

It is not difficult to see that

$$F(m) = 2^{n-1} F_+(m), \qquad (4)$$

$$\sum_d F_+\left(\frac{m}{d}\right) = m^{n+1}, \qquad (5)$$

where the sum is taken over all divisors of m.

From (5) it follows that

$$\left(\sum_{d=1}^{\infty} \frac{1}{d^s}\right)\left(\sum_{m=1}^{\infty} \frac{F_+(m)}{m^s}\right) = \sum_{m=1}^{\infty} \frac{m^{n+1}}{m^s} = \zeta(s - n + 1), \qquad (6)$$

where $\zeta(s)$ is the Riemann Zeta-function.

From (4) and (6) it follows that

$$D(s) = 2^{n-1} \frac{\zeta(s - n + 1)}{\zeta(s)}. \qquad (7)$$

§ 2. ANALYSIS ON THE GROUP OF ADELES

1. Schwartz-Bruhat Functions. In this subsection we introduce a space of functions on the adele group A, which is important for

what follows. As we shall see in the next subsection, this space is invariant under Fourier transformations.

On the adele group we consider functions $\varphi(a)$ that are representable in the form of an infinite product

$$\varphi(a) = \prod_p \varphi_p(a_p), \tag{1}$$

where the factors $\varphi_p(a_p)$ satisfy the following conditions:

1. $\varphi_\infty(a_\infty)$ is an infinitely differentiable function on R, that is, on the additive group of real numbers, and decreases faster than any power of $|a_\infty|$ as $|a_\infty| \to \infty$.

2. $\varphi_p(a_p)$, $p = 2, 3, \ldots$ is finite and piecewise constant, that is, constant on the cosets of a sufficiently small open subgroup of the group of p-adic numbers Q_p.

3. For all p, except a finite number, $\varphi_p(a_p) = 1$, when a_p is a p-adic integer, and $\varphi_p(a_p) = 0$, when a_p is not an integer.

By 3, for every $a \in A$ all the factors in the infinite product (1) from a certain p onward are equal to 1. Thus, the product converges. Hence, it follows easily that $\varphi(a)$ is a continuous function on A.

Note that by the same condition 3 the function $\varphi(a)$ is concentrated on an open subgroup of A of the form

$$A_\infty \times A_2 \times \cdots \times A_p \times V_p,$$

where V_p is the subgroup of adeles for which all the components are p-adic integers, and $a_\infty = a_2 = \cdots a_p = 0$, where p is sufficiently large. Moreover, the function is constant on the cosets of V_p. Hence, it can be regarded as a function on the group

$$A^{(p)} = A_\infty \times {}_2 \times \cdots \times A_p,$$

where p is a sufficiently large integer. The same remark applies to any finite linear combination of functions of the form (1).

We call functions of the form (1) elementary functions on A.

We use the term *Schwartz-Bruhat functions*† for functions $\varphi(a)$ on A that are representable as finite linear combinations of elementary functions. We denote by $S(A)$ the set of all Schwartz-Bruhat functions on A.

It is not hard to check that all the functions $\varphi(a) \in S(A)$ are summable on A, that is, that for every function $\varphi \in S(A)$

$$\int_A |\varphi(a)|\, da < \infty.$$

2. The Fourier Transform of Schwartz-Bruhat Functions. Let $\chi_0(a)$ be a character on A defined in **§ 1.5**:

$$\chi_0(a) = \exp 2\pi i \sigma(a), \tag{1}$$

† The name was introduced by Godement who apparently was first to recognize the important role of these functions on the adele group.

where

$$\sigma(a) \equiv -a_\infty + a_2 + \cdots + a_p + \cdots \pmod 1.$$

We define the Fourier transform of $\varphi(a)$ by the formula

$$\tilde{\varphi}(b) = \int \varphi(a)\chi_0(ba)\,da. \tag{2}$$

Although we make no use of it in what follows, we mention that the Fourier transform can be defined for every measurable function $\varphi(a)$ of integrable square modulus; the integral (2) must then be understood in the sense of the mean square value.

By the formula for the inverse Fourier transform we have

$$\varphi(a) = c\int \tilde{\varphi}(b)\chi_0(-ba)\,db, \tag{3}$$

in other words,

$$\tilde{\tilde{\varphi}}(-a) = c^{-1}\varphi(a). \tag{4}$$

Next, the Plancherel formula holds:

$$\int |\varphi(a)|^2\,da = c\int |\tilde{\varphi}(a)|^2\,da. \tag{5}$$

Here c is a constant depending on the normalization of the measure on A. It is easy to check that under the normalization of da that was introduced at the very beginning, we have

$$c = 1.$$

Let us show that *the Fourier transform of a function in* $S(A)$ *is again a function in* $S(A)$.

For this purpose it is sufficient to show that if $\varphi(a)$ is an elementary function, that is, a function of the form (1), where $\varphi_p(a_p)$ satisfies the conditions 1 through 3 of **§ 2.1**, its Fourier transform $\tilde{\varphi}(a)$ is a function of the same form.

First, it is evident that

$$\tilde{\varphi}(b) = \prod_p \tilde{\varphi}_p(b_p),$$

where

$$\tilde{\varphi}_p(b_p) = \int \varphi_p(a_p)\chi_0(b_p a_p)\,da_p, \qquad p = \infty, 2, 3, \ldots.$$

Now it is known that the Fourier transform of an infinitely differentiable function $\varphi_\infty(a_\infty)$ decreasing faster than any power of $|a_\infty|$, as $|a_\infty| \to \infty$, is a function of the same form. Consequently condition 1 holds for $\tilde{\varphi}(b)$.

Next, the Fourier transform of the finite piecewise constant function $\varphi_p(a_p)$, $p = 2, 3, \ldots$ is a function of the same form (see Chapter 2, **§ 2.4**). Consequently, condition 2 holds for $\tilde{\varphi}(b)$.

Finally, we use the fact that the Fourier transform carries a function of the form

$$\varphi_p(a_p) = \begin{cases} 1 & \text{for} \quad |a_p|_p \leq 1, \\ 0 & \text{for} \quad |a_p|_p > 1 \end{cases} \tag{6}$$

into itself (see Chap. II, **§ 2.4**). Hence, it follows that condition 3 holds for $\tilde{\varphi}(b)$.

From the formula $\tilde{\tilde{\varphi}}(-a) = \varphi(a)$ it follows easily that the Fourier transform maps $S(A)$ onto itself.

3. The Poisson Summation Formula. Let $\varphi(a)$ be a Schwartz-Bruhat function and $\tilde{\varphi}(a)$ its Fourier transform. We shall derive the following formula, which is usually called the *Poisson summation formula:*

$$\sum_{\alpha \in Q} \varphi(\lambda\alpha) = \frac{1}{|\lambda|} \sum_{\alpha \in Q} \tilde{\varphi}(\lambda^{-1}\alpha), \tag{1}$$

where λ is an arbitrary idele and the summation is taken over the subgroup Q of principal ideles.†

To prove formula (1) we introduce an auxiliary function on the group of adeles:

$$\Phi(a) = \sum_{\alpha \in Q} \varphi(\lambda(\alpha + a)), \tag{2}$$

where the summation is taken over the subgroup of principal adeles.

From the summability of $\varphi(a)$ it follows immediately that the series (2) converges absolutely almost everywhere and that $\Phi(a)$ is a summable function on A/Q. In fact,

$$\int_A |\varphi(\lambda a)|\, da = \int_{A/Q} \left(\sum_{\alpha \in Q} |\varphi(\lambda(\alpha + a))|\right) da.$$

It is also easy to verify that $\Phi(a)$ is a continuous function.

Since $\Phi(a)$ is constant on the cosets of Q and summable on the compact group A/Q, it can be expanded as a Fourier series with respect to the characters of A that are equal to 1 on Q. As was shown in **§ 1.6**, these characters are of the form $\chi_0(\beta a)$, where β ranges over the principal adeles. So we have

$$\Phi(a) = \sum_{\beta \in Q} c_\beta \chi_0(\beta a). \tag{3}$$

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† The Poisson formula holds for every commutative topological group G with a discrete subgroup Γ and a compact factor group G/Γ. The classical Poisson formula corresponds to the case when G is the group of all real numbers, and Γ the subgroup of integers.

The Fourier coefficients c_β can be expressed by the following formula:

$$c_\beta = \int_{A/Q} \Phi(a)\chi_0(-\beta a)\, da. \tag{4}$$

When we substitute for $\Phi(a)$ its expression (2), we obtain

$$\begin{aligned} c_\beta &= \int_{A/Q} \Big(\sum_{a\in Q} \varphi(\lambda(\alpha + a))\Big)\chi_0(-\beta a)\, da \\ &= \int_A \varphi(\lambda a)\chi_0(-\beta a)\, da = \frac{1}{|\lambda|}\int_A \varphi(a)\chi_0(-\beta\lambda^{-1}a)\, da \\ &= \frac{1}{|\lambda|}\tilde{\varphi}(-\beta\lambda^{-1}). \end{aligned}$$

So we have

$$\sum_{a\in Q} \varphi(\lambda(\alpha + a)) = \frac{1}{|\lambda|}\sum_{\beta\in Q}\tilde{\varphi}(-\lambda^{-1}\beta)\chi_0(\beta a). \tag{5}$$

Hence, for $a = 0$ we obtain the Poisson formula (1).

4. The Mellin Transform of Schwartz-Bruhat Functions. The Tate Formula. Let $\varphi(a)$ be a Schwartz-Bruhat function on the group of adeles A. Since the idele group A^* can be embedded, in a unique fashion, as a subset in the group of adeles A, we can investigate the restriction $\varphi(\lambda)$ of φ to A^*.

Let $\pi(\lambda)$ be a character on the group of adeles A^* that is identically equal to 1 on the subgroup Q^* of principal ideles. We use the term *Mellin transform* of φ for the function $\Phi(\pi)$ defined by the following formula:

$$\Phi(\pi) = \int_{A^*} \varphi(\lambda)\pi(\lambda)\, d^*\lambda, \tag{1}$$

where $d^*\lambda$ is the invariant measure on the group of ideles.

Let us find out for what characters π the integral (1) converges.

We recall that every character π can be represented in the form

$$\pi(\lambda) = |\lambda|^s\,\theta(\lambda), \tag{2}$$

where

$$|\lambda| = |\lambda_\infty|_\infty\, |\lambda_2|_2 \cdots |\lambda_p|_p, \ldots,$$

s is a complex number, and $|\theta(\lambda)| = 1$; the number s and the character θ are uniquely determined by the formulae (5) and (6) of **§ 1.10**. So we can write

$$\Phi(\pi) \equiv \Phi(\theta, s) = \int_{A^*} \varphi(\lambda)\theta(\lambda)\, |\lambda|^s\, d^*\lambda. \tag{3}$$

The question is, therefore, for what s the integral (3) converges.

In answering this question, we may confine ourselves to elementary functions, that is, functions of the form

$$\varphi(\lambda) = \varphi_\infty(\lambda_\infty)\varphi_2(\lambda_2)\cdots\varphi_p(\lambda_p)\cdots.$$

Since

$$|\lambda| = |\lambda_\infty|_\infty\,|\lambda_2|_2\cdots|\lambda_p|_p\cdots$$

and

$$\theta(\lambda) = \theta_\infty(\lambda_\infty)\theta_2(\lambda_2)\cdots\theta_p(\lambda_p)\cdots,$$

the integral (3) can be rewritten as an infinite product of integrals:

$$\Phi(\theta, s) = \prod_p \int_{Q_p^*} \varphi_p(\lambda_p)\theta_p(\lambda_p)\,|\lambda_p|_p^s\,d^*\lambda_p. \tag{4}$$

Evidently each factor of this product is an absolutely convergent integral for $\operatorname{Re} s > 0$. The question is: Under what additional conditions on s does the infinite product converge? Here we observe that by the definition of Schwartz-Bruhat functions the $\varphi_p(\lambda_p)$ are concentrated for sufficiently large λ_p on the set of integers λ_p and are equal to 1 on this set. On the other hand, for sufficiently large p the characters θ_p have the following form:

$$\theta_p(\lambda_p) = |\lambda_p|_p^{is_p},$$

where s_p is a real number.

Thus, for sufficiently large p we have†

$$\int_{Q_p^*} \varphi(\lambda_p)\theta_p(\lambda_p)\,|\lambda_p|_p^s\,d^*\lambda_p = \int_{|\lambda_p|_p \le 1} |\lambda_p|_p^{s+is_p}\,d^*\lambda_p = 1 + \frac{p^{-(s+is_p)}}{1 - p^{-(s+is_p)}}. \tag{5}$$

It is easy to verify that the infinite product

$$\prod_p\left(1 + \frac{p^{-(s+is_p)}}{1 - p^{-(s+is_p)}}\right)$$

converges for $\operatorname{Re} s > 1$.

So we have shown that the integral (3) converges absolutely for $\operatorname{Re} s > 1$.

From (4) and (5) it follows that $\Phi(\theta, s)$ is an analytic function of s in the domain $\operatorname{Re} s > 1$ for fixed φ and θ.

We shall now show that *the function* $\Phi(\theta, s)$ *has an analytic continuation to the whole plane of the complex variable* s. *Its only singularities are*

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† We recall that the measure $d^*\lambda_p$ on Q_p^* induced by $d^*\lambda$ is normalized by the condition $\int_{|\lambda_p|_p=1} d^*\lambda_p = 1$.

simple poles at the points $s = 0$ and $s = 1$. The residues of $\Phi(\theta, s)$ at these poles are $-\varepsilon_\theta\phi(0)$ and $\varepsilon_\theta\tilde{\phi}(0)$, respectively, where $\varepsilon_\theta = 1$ for $\Theta \equiv 1$, and $\varepsilon_\theta = 0$ otherwise.

Proof. We split $\Phi(\theta, s)$ into the sum of two integrals

$$\Phi(\theta, s) = \Phi^+(\theta, s) + \Phi^-(\theta, s), \tag{6}$$

where

$$\Phi^+(\theta, s) = \int_{|\lambda|\geq 1} \varphi(\lambda)\theta(\lambda)\, |\lambda|^s\, d^*\lambda, \tag{7}$$

$$\Phi^-(\theta, s) = \int_{|\lambda|\leq 1} \varphi(\lambda)\theta(\lambda)\, |\lambda|^s\, d^*\lambda. \tag{8}$$

We have shown above that $\Phi(\theta, s)$ is absolutely convergent for $\operatorname{Re} s > 1$ and is an analytic function of s in the domain $\operatorname{Re} s > 1$. The same is true for the integrals $\Phi^+(\theta, s)$ and $\Phi^-(\theta, s)$.

Now we observe that $\Phi^+(\theta, s)$ must also converge for $\operatorname{Re} s \leq 1$ and in this domain is an analytic function of s; therefore, $\Phi^+(\theta, s)$ is an entire analytic function of s.

Thus, to prove the theorem we consider the second integral $\Phi^-(\theta, s)$.

Let us transform this integral. Since $|\alpha| = 1$ for every principal adele α, the set $|\lambda| \leq 1$ is invariant under the discrete group Q^* of transformations

$$\lambda \to \lambda\alpha$$

where α ranges over the principal ideles. Let E be a fundamental domain in the set $|\lambda| \leq 1$ relative to the discrete group Q^*. Then we have

$$\Phi^-(\theta, s) = \int_E \sum_{\alpha\in Q^*} \varphi(\lambda\alpha)\theta(\lambda)\, |\lambda|^s\, d^*\lambda. \tag{9}$$

(Here we use the fact that $\theta(\alpha) = 1$.)

We apply the Poisson summation formula:

$$\sum_{\alpha\in Q} \varphi(\lambda\alpha) = \frac{1}{|\lambda|} \sum_{\alpha\in Q} \tilde{\varphi}(\lambda^{-1}\alpha), \tag{10}$$

where $\tilde{\varphi}$ is the Fourier transform of φ. Note that the set Q^* of principal ideles is obtained from the set Q of principal adeles by omitting the element 0; consequently, (10) can be rewritten in the following form:

$$\varphi(0) + \sum_{\alpha\in Q^*} \varphi(\lambda\alpha) = \frac{1}{|\lambda|}\, \tilde{\varphi}(0) + \frac{1}{|\lambda|} \sum_{\alpha\in Q^*} \tilde{\varphi}(\lambda^{-1}\alpha) \tag{11}$$

When we substitute in (9) for $\sum_{\alpha \in Q^*} \varphi(\lambda\alpha)$ its expression from (11), we obtain

$$\begin{aligned}\Phi^-(\theta, s) &= \int_E \sum_{\alpha \in Q^*} \tilde{\varphi}(\lambda^{-1}\alpha)\theta(\lambda)\, |\lambda|^{s-1}\, d^*\lambda \\ &\quad + \tilde{\varphi}(0)\int_E \theta(\lambda)\, |\lambda|^{s-1}\, d^*\lambda - \varphi(0)\int_E \theta(\lambda)\, |\lambda|^s\, d^*\lambda \\ &= \int_E \sum_{\alpha \in Q^*} \tilde{\varphi}(\lambda\alpha)\theta^{-1}(\lambda)\, |\lambda|^{1-s}\, d^*\lambda \\ &\quad + \tilde{\varphi}(0)\int_E \theta(\lambda)\, |\lambda|^{s-1}\, d^*\lambda - \varphi(0)\int_E \theta(\lambda)\, |\lambda|^s\, d^*\lambda. \qquad (12)\end{aligned}$$

The first term in this formula is $\tilde{\Phi}^+(\theta^{-1}, 1 - s)$, where $\tilde{\Phi}$ is the Mellin transform of $\tilde{\varphi}$.

Now we evaluate the integral $\int_E \theta(\lambda)\, |\lambda|^s\, d^*\lambda$ in (12). Clearly, the integral is zero when $\theta(\lambda) \not\equiv 1$. Now let $\theta(\lambda) \equiv 1$. For the fundamental domain E we choose the set of ideles

$$\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots),$$

where $0 < \lambda_\infty < 1$ and $|\lambda_p|_p = 1$ (see **§ 2.10**). Then we have

$$\int_E |\lambda|^s d^*\lambda = \int_0^1 \lambda_\infty^{s-1}\, d\lambda_\infty \int_{|\lambda_2|_2=1} d^*\lambda_2 \cdots \int_{|\lambda_p|_p=1} d^*\lambda_p \cdots$$

Since $\int_{|\lambda_p|_p=1} d^*\lambda_p = 1$, we now obtain that

$$\int_E |\lambda|^s\, d^*\lambda = \frac{1}{s}.$$

Thus, finally we have

$$\int_E \theta(\lambda)\, |\lambda|^s\, d^*\lambda = \frac{\varepsilon_\theta}{s}, \qquad (13)$$

where $\varepsilon_\theta = 1$ when $\theta(\lambda) \equiv 1$, and $\varepsilon_\theta = 0$, when $\theta(\lambda) \not\equiv 1$.

So we have reached the following equation:

$$\Phi^-(\theta, s) = \tilde{\Phi}^+(\theta^{-1}, 1 - s) + \varepsilon_\theta\left(\frac{\tilde{\varphi}(0)}{s - 1} - \frac{\varphi(0)}{s}\right), \qquad (14)$$

where $\tilde{\Phi}$ denotes the Mellin transform of $\tilde{\varphi}$.

As we know already, Φ^+ is an entire analytic function of s. Thus, by (14) $\Phi^-(\theta, s)$ is an analytic function of s on the whole complex s-plane and its only singularities are simple poles at the

point $s = 0$ and $s = 1$. The residues of Φ^- at these points are $-\varepsilon_\theta \varphi(0)$ and $\varepsilon_\theta \tilde{\varphi}(0)$, respectively. The proposition is now proved.

A consequence of (12) is the following functional relation for $\Phi(\theta, s)$ (*Tate's formula*):

$$\Phi(\theta, s) = \tilde{\Phi}(\theta^{-1}, 1 - s), \tag{15}$$

where $\tilde{\Phi}$ is the Mellin transform of the function $\tilde{\varphi}(a)$.

Indeed, when we replace in (14) $\varphi(a)$, θ and s by $\tilde{\varphi}(a)$, θ^{-1} and $1 - s$, respectively, we find

$$\tilde{\Phi}^-(\theta^{-1}, 1 - s) = \Phi^+(\theta, s) - \varepsilon_\theta\left(\frac{\varphi(0)}{s} - \frac{\tilde{\varphi}(0)}{s - 1}\right).$$

hence,

$$\Phi^+(\theta, s) = \tilde{\Phi}^-(\theta^{-1}, 1 - s) - \varepsilon_\theta\left(\frac{\tilde{\varphi}(0)}{s - 1} - \frac{\varphi(0)}{s}\right). \tag{16}$$

By adding (14) and (16) term-by-term we obtain the Tate formula (15).

As a consequence of the Tate formula we now obtain the functional relation for the Riemann Zeta-function $\zeta(s)$.

For this purpose we consider the function $\varphi(a)$ of the form

$$\varphi(a) = \varphi_\infty(a_\infty)\varphi_2(a_2) \cdots \varphi_p(a_p), \ldots,$$

where

$$\varphi_\infty(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right),$$

$$\varphi_p(a_p) = \begin{cases} 1, & \text{when } |a_p|_p \le 1, \\ 0, & \text{when } |a_p|_p > 1. \end{cases}$$

It is known that $\tilde{\varphi}_\infty(x) = \varphi_\infty(x)$. On the other hand, as was shown in Chapter 2, **§ 2.4**, $\tilde{\varphi}_p(a_p) = \varphi_p(a_p)$ for every p. Consequently,

$$\tilde{\varphi}(a) = \varphi(a). \tag{17}$$

Let us compute $\Phi(\theta_0, s)$ for the function φ, when $\theta_0(\lambda) \equiv 1$. We have

$$\Phi(\theta_0, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2} |x|^{s-1}\, dx \prod_p \int_{|\lambda_p|_p \le 1} |\lambda_p|_p^s \, d^*\lambda_p. \tag{18}$$

All the integrals in (18) can be computed directly. We have

$$\int_{|\lambda_p|_p \le 1} |\lambda_p|_p^s \, d^*\lambda_p = \frac{1}{1 - p^{-s}}.$$

Consequently,

$$\prod_p \int_{|\lambda_p|_p \le 1} |\lambda_p|_p^s \, d^*\lambda_p = \prod_p \frac{1}{1 - p^{-s}} = \zeta(s),$$

where $\zeta(s)$ is the Riemann Zeta-function. On the other hand, we have

$$\int_{-\infty}^{+\infty} e^{-x^2/2} |x|^{s-1}\, dx = 2^{s/2}\Gamma\left(\frac{s}{2}\right),$$

where $\Gamma(x)$ is the classical Gamma-function.

Thus,

$$\Phi(\theta_0, s) = \frac{1}{\sqrt{2\pi}} 2^{s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{19}$$

Therefore, the Tate formula (15) gives us the required relation for the Zeta-function:

$$2^{s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = 2^{1-s/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{20}$$

Similarly, we can derive from the Tate formula the functional relation for the Dirichlet L-function.

5. The Space A^n. Now we investigate the n-dimensional vector space over the group of adeles A, that is, the space of points

$$y = (y^{(1)}, \ldots, y^{(n)}), \tag{1}$$

where

$$y^{(i)} = (y^{(i)}_\infty, y^{(i)}_2, \ldots, y^{(i)}_p, \ldots) \tag{2}$$

are elements of A. We denote this space by A^n.

By analogy with the group A we introduce the concept of a Schwartz-Bruhat function on A^n and that of a Fourier transform on A^n. Then we study the Mellin transform in A^n.

We consider a function $\varphi(y)$ on A^n of the form

$$\varphi(y) = \varphi_\infty(y_\infty)\varphi_2(y_2) \cdots \varphi_p(y_p), \ldots, \tag{3}$$

where

$$y_p = (y^{(1)}_p, \ldots, y^{(n)}_p), \qquad p = \infty, 2, 3, \ldots,$$

that satisfies the following conditions:

1. $\varphi_\infty(y_\infty)$ is an infinitely differentiable function in the n-dimensional space R^n, over the field of real numbers and decreasing faster than any power of $|y_\infty|$ as $|y_\infty| \to \infty$, where $|y_\infty|$ is the norm of the vector y_∞:

$$|y_\infty| = (|y^{(1)}_\infty|^2 + \cdots + |y^{(n)}_\infty|^2)^{1/2}.$$

2. The function $\varphi_p(y_p)$ is finite and piecewise constant, $p = 2, 3, \ldots$

3. For all p except a finite number the function $\varphi_p(y_p)$ is concentrated on the set of vectors

$$y_p = (y^{(1)}_p, \ldots, y^{(n)}_p)$$

whose coordinates are p-adic integers, and is identically equal to one on this set.

Functions of the form (3) satisfying the conditions 1 through 3 are called *elementary functions* on A^n. *Schwartz-Bruhat functions* on A^n are defined as functions that can be represented as finite linear combinations of elementary functions. The space of Schwartz-Bruhat functions is denoted by $S(A^n)$.

It is easy to verify that Schwartz-Bruhat functions are continuous and summable on A^n with respect to the measure $dy = dy^{(1)} \cdots dy^{(n)}$.

We define the Fourier transform of a function $\varphi \in S(A^n)$ by the following formula:

$$\tilde{\varphi}(\xi) = \int \varphi(y)\chi_0(y \cdot \xi)\, dy, \tag{4}$$

where $\xi \in A^n$,

$$y \cdot \xi = y^{(1)}\xi^{(1)} + \cdots + y^{(n)}\xi^{(n)},$$
$$dy = dy^{(1)} \cdots dy^{(n)},$$

and $\chi_0(a) = \exp 2\pi i \sigma(a)$ is the character defined in **§ 2.4.** The following result holds.

The Fourier transformation carries a function in $S(A^n)$ into a function in $S(A^n)$; in fact, it maps the space $S(A^n)$ onto itself.

The proof is almost identical to that of the corresponding result for functions in $S(A)$ (see **§ 2.2**).

We derive the Poisson summation formula for the space A^n:

$$\sum_{\alpha \in Q^n} \varphi(\lambda\alpha) = \frac{1}{|\lambda|^n} \sum_{\alpha \in Q^n} \tilde{\varphi}(\lambda^{-1}\alpha), \tag{5}$$

where λ is an arbitrary idele; the summation is taken over the vectors $\alpha \in A^n$ whose coordinates are all principal adeles.

The derivation of this formula proceeds word for word just as in the one-dimensional case (see **§ 2.3**).

Next, we introduce the concept of the Mellin transform of a function $\varphi \in S(A^n)$.

The Mellin transform of a Schwartz-Bruhat function $\varphi(y)$ on A^n is defined by the following formula:

$$\Phi(y; \pi) = \int_{A^*} \varphi(\lambda y)\pi(\lambda)\, d^*\lambda, \qquad y \neq 0, \tag{6}$$

where $\pi(\lambda)$ is a character on the group of ideles A^* that is identically equal to 1 on the subgroup Q^* of principal ideles, and $d^*\lambda$ is the invariant measure on A^*.

We know (see **§ 1.10**) that the character π has the form

$$\pi(\lambda) = |\lambda|^s\, \theta(\lambda), \tag{7}$$

where s is a complex number, and θ is a character given initially on a compact subgroup Λ of A^* and extended to the whole group A^*.

Thus, Φ is a function of s and θ, where θ ranges over a discrete set.

From the results in **§ 1.10** it follows that the integral (6) converges for $\operatorname{Re} s > 1$ and is in this domain an analytic function of s. The function $\Phi(y; \pi) \equiv \Phi(y; \theta, s)$ has an analytic continuation to the whole complex s-plane. Its only singularities are simple poles at $s = 0$ and $s = 1$. The residues of Φ at these points are $-\varepsilon_\theta \varphi(0)$ and $\varepsilon_\theta \int_A \varphi(\lambda y)\, d\lambda$, respectively, where $\varepsilon_\theta = 1$, when $\theta \equiv 1$, and $\varepsilon_\theta = 0$ otherwise (the integral is taken over the group of adeles A).

Now we mention a property of Φ as a function of y. From (6) it follows immediately that

$$\Phi(\lambda y, \pi) = \pi^{-1}(\lambda)\Phi(y, \pi) \tag{8}$$

for every idele $\lambda \in A^*$ (the property of homogeneity of Φ).

Since $\pi(\lambda) = 1$ on the subgroup Q^* of principal ideles, we have by (8)

$$\Phi(\lambda y; \pi) = \Phi(y, \pi)$$

for every principal idele $\lambda \in Q^*$. Thus, Φ can be regarded as a function on the factor space $\Omega = Q^* \setminus A^n$, obtained from A^n by identifying y and λy, where $\lambda \in Q^*$.

APPENDIX TO § 2

Tate Rings

Let A be a commutative locally compact ring. By A' we denote the group of characters of the additive group of A.

The map

$$\chi(a) \to \chi(ra),$$

where r is any element of A, defines an endomorphism of A'.

As we mentioned in **§ 1**, A' is a module over A. We recall that the ring A is called self-dual if the module A' is isomorphic to A.

We denote by A^* the set of all elements of A that have an inverse element. Generally speaking, A^* is not a closed subset of A. However, we can introduce a topology in A^* under which it becomes a topological group. Specifically, a neighborhood U of an element $a_0 \in A^*$ consists of those elements $a \in A^*$ for which $a \in V(a_0)$ and $a^{-1} \in W(a_0^{-1})$, where V and W are given neighborhoods in A of a_0 and a_0^{-1}, respectively.

It is not hard to verify that under this topology A^* becomes a locally compact topological group, when the multiplication of elements is taken as group operation.

Next, we denote by da a measure on A invariant under addition and by d^*a a measure on A^* invariant under multiplication. For every element $\alpha \in A^*$ we define its norm in the following way. We examine the measure $d_\alpha(a) = d(\alpha a)$. It is not hard to see that this measure is invariant under addition, and hence, is proportional to da. We set

$$|\alpha| = \frac{d(\alpha a)}{da}. \tag{1}$$

For elements $\alpha \notin A^*$ the norm is not defined.

Let Q be a subring of A. We call Q unimodular if the norm of each nonzero element is 1. Obviously, a unimodular ring is a field and is discrete.

Let A be a self-dual ring with a distinguished unimodular subring Q.

The pair (A, Q) is called a *Tate pair* if the following conditions are satisfied:

1. The subgroup of additive characters on A that are equal to 1 on Q is isomorphic to Q.

2. The group A_1^*/Q^* is compact, where A_1^* is the group of elements of norm 1, and Q^* is the multiplicative group of Q.

According to the Pontryagin duality theorem it follows from 1 that A/Q is the character group of Q. Consequently, since Q is discrete, A/Q is compact.

Now we write down the Poisson formula. Let $\varphi(a)$ be a function on A for which the series $\psi(a) = \sum_{\alpha \in Q} \varphi(a + \alpha)$ converges absolutely and uniformly and, in addition, $\psi(a)$ can be expanded in an absolutely convergent Fourier series. Then

$$\sum_{\alpha \in Q} \varphi(a) = \sum_{\chi \in (A/Q)'} \int_A \varphi(a)\bar{\chi}(a)\, da, \tag{2}$$

where χ ranges over all characters of A that are equal to 1 on Q. For the proof we consider the expansion of the function $\psi(a) = \sum_\alpha \varphi(a + \alpha)$ in a Fourier series (see **§ 2.3**).

Now we go on to the discussion of the Mellin transform. We denote by Π the set of all characters† of A^* that are equal to 1 on $Q^* = A^* \cap Q$.

† Here by a *character* we mean an arbitrary solution (not necessarily of mod 1) of the functional equation

$$\varphi(ab) = \varphi(a)\varphi(b), \quad a, b \in A^*.$$

Let φ be a function on A. Its Mellin transform is defined as the integral ($\pi \in \Pi$)

$$\Phi(\pi) = \int_{A^*} \varphi(a)\pi(a)\, d^*a. \tag{3}$$

By L we denote the set of functions $\varphi(a)$ defined on A and having the following properties:

1. The series $\psi(a) = \sum_{\alpha \in Q} \varphi(a + \alpha)$ converges absolutely and uniformly.

2. $\psi(a)$ can be expanded in an absolutely convergent Fourier series with respect to the (additive) characters of A/Q.

3. The integrals $\int \varphi(a)\, |a|^s\, d^*a$ and $\int \tilde{\varphi}(a)\, |a|^s\, d^*a$ converge absolutely for all sufficiently large values of Re s.

Under these assumptions a repetition of the arguments in § 2.4 shows that for every function $\varphi(a) \in L$ its Mellin transform has an analytic continuation to the whole of Π, that its only poles are at the points $\pi(a) \equiv 1$ and $\pi(a) \equiv |a|$, and that it satisfies the functional equation

$$\Phi(\pi, \varphi) = \Phi(\tilde{\pi}, \tilde{\varphi}), \qquad \tilde{\pi}(a) = |a|\, \pi^{-1}(a).$$

§ 3. THE GROUPS OF ADELES G_A AND THEIR REPRESENTATIONS

1. Definition of the Group of Adeles G_A. The concepts of adeles and ideles, which were introduced by Chevalley with algebraic number theory in mind, have proved to be useful when generalized to the case of an arbitrary linear algebraic group defined over the field of rational numbers Q. This generalization, which was proposed by A. Weil, consists in the following:

Let G be a linear algebraic group† defined over the field of rational numbers Q.

To specify G it is sufficient to know the set of polynomial relations among the elements of the matrices of G. We denote by G_p the set of all p-adic matrices belonging to G, and by U_p the integral subgroup of G_p (that is, the subgroup of matrices g_p for which the elements of g_p and of g_p^{-1} are p-adic integers). By G_∞ we denote the set of all real matrices belonging to G. We consider infinite sequences

$$g = (g_\infty, g_2, \ldots, g_p, \ldots), \qquad g_p \in G_p, \tag{1}$$

where all the g_p except a finite number belong to U_p. Such sequences

† For the definition of a linear algebraic group see the Appendix to Chapter 1.

are called *adeles* of G. They form a group G_A. (Multiplication is componentwise.)

A topology in G_A is introduced in the following way. We take the subgroup G_A^0 of adeles $(g_\infty, g_2, \ldots, g_p, \ldots)$ where $g_p \in U_p$ for every prime p. In G_A^0 we introduce the topology of the Tikhonov product of the topological spaces G_∞, U_2, $\ldots$, U_p, $\ldots$. This subgroup G_A^0 is declared to be an open set in G_A. The topological group G_A so obtained is called the group of *adeles* of the given group G.

The group of adeles is locally compact; this follows immediately from the fact that U_p is compact and G_∞ is locally compact.

The group G_Q can be isomorphically embedded in the group of adeles G_A. For Q is embedded in the field of real numbers R as well as the field Q_p of p-adic numbers. Therefore G_q is isomorphically embedded in G_∞ as well as in G_p. With every element $r \in G_Q$ we associate the sequence

$$(r, r, \ldots, r, \ldots). \tag{2}$$

It is easy to verify that these sequences are adeles. (This follows from the fact that any rational number is a p-adic integer for sufficiently large p.) They are called *principal adeles* of the given group G.

Let us show that *the subgroup of principal adeles* $\Gamma = G_Q$ *is discrete in* G_A.

Proof. Suppose that G_Q is not discrete. Then we can find a sequence of principal adeles $(r_n, r_n, \ldots, r_n, \ldots)$, converging to the unit element of G_A. From the definition of the topology in G_A it follows then that beginning with a certain n, the elements of the matrices r_n are p-adic integers for every p. But a rational number is a p-adic integer for every p if and only if it is an integer. So we have shown that the elements of the matrices r_n are integers from a certain n onward. Hence, it follows that a sequence of pairwise distinct matrices r_n cannot converge in the topology of G_∞; and so we have arrived at a contradiction.

2. Irreducible Unitary Representations of the Group of Adeles. In this and the next subsection we show how the classification of all unitary representations of the group of adeles G_A reduces to the classification of the unitary representations of group G_p, $p = \infty$, 2, 3, ... Specifically, we show that under some conditions on G every irreducible unitary representation of G_A is given in the following way:

Suppose that for each p an irreducible unitary representation $T_p(g_p)$ of G_p is given in a Hilbert space H_p. We assume that for all sufficiently large p the representation spaces H_p contain at least

one vector invariant under U_p. Such representations T_p are called *representations of class* I.

We choose in H_p an orthonormal basis ξ_p^k, $k = 1, 2, \ldots$ and denote by ξ_p^1 a vector invariant under U_p, if such a vector exists in H_p.

Now we take another Hilbert space in which an orthonormal basis is given by the formal products $\xi = \bigotimes_p \xi_p^i p$, where in each product $i_p = 1$ for all p except a finite number. H is, of course, a separable space.

The representation operator is given by the formula

$$T(g)\,\xi = \prod_p T_p(g_p)\,\xi_p^{i_p}, \tag{1}$$

where

$$g = (g_\infty, g_2, \ldots, g_p, \ldots); \qquad \xi = \bigotimes_p \xi_p^{i_p}.$$

The resulting representation of the group is called the *tensor product* of the representations $T_p(g_p)$ of G_p.

We remark that the construction of the tensor representation depends not only on the choice of the sequence of the unitary representations H_p, but also on the choice in each H_p of a vector invariant under U_p. Clearly, choices differing only at a finite number of places lead to equivalent representations; choices differing at an infinite number of places lead to inequivalent representations. Also, for many important groups, for example when G is a Dickson-Chevalley group, it can be shown that H_p contains not more than one linearly independent vector invariant under U_p. It is very likely that the corresponding statement is true for every reductive linear algebraic group. However, we do not know whether this has been proved yet.

The construction described above for a representation of G_A in H always leads to an irreducible representation. This can be proved by means of standard devices in the theory of representations, and we shall not dwell on it here.

We give, also without proof, the formula for the character of $T(g)$. It is well known that the character of the tensor product of two representations $T_1(g_1)$ and $T_2(g_2)$ is equal to the product of the characters $\mathrm{Tr}\ T_1(g_1)$ and $\mathrm{Tr}\ T_2(g_2)$ of these representations. A similar result holds in our situation: the character $\mathrm{Tr}\ T(g)$ of $T(g)$ is equal to

$$\mathrm{Tr}\ T(g) = \prod_p \mathrm{Tr}\ T_p(g_p),$$

where $\mathrm{Tr}\ T_p(g_p)$ is a character of $T_p(g_p)$. Here $\mathrm{Tr}\ T(g)$ must be understood as a generalized function, that is, as a functional on a suitable chosen family of functions on G_A.

3. Proof of a Theorem on Tensor Products. The main result of the present section is the following theorem.

We assume that all the groups G_p are of type I and satisfy for all but finitely many p the following condition:† each irreducible representation of G_p contains not more than one vector invariant under a maximal compact subgroup.

Then every irreducible unitary representation of G_A is a tensor product (in the sense of **§ 3.2**) of irreducible unitary representations of the G_p, and all the T_p except a finite number are representation of class I.

First we prove two lemmas.

LEMMA 1. *Suppose that a locally compact topological group $\mathfrak{G}$ is the direct product of two subgroups,*

$$\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2,$$

where at least one of the factors, say $\mathfrak{G}_1$, is a group of type I. *Then every irreducible unitary representation of $\mathfrak{G}$ is a tensor product of irreducible representations of $\mathfrak{G}_1$ and $\mathfrak{G}_2$.*

Proof. Let $T(g)$ be an irreducible unitary representation of $\mathfrak{G}$ acting in a Hilbert space H. We consider the weakly-closed rings R_1 and R_2 of bounded operators in H that are generated, respectively, by the operators $T(g_1)$, $g_1 \in \mathfrak{G}_1$ and $T(g_2)$, $g_2 \in \mathfrak{G}_2$. Let R_i' be the ring of bounded operators that commute with all the operators R_i, $i = 1, 2$.

Since the elements of $\mathfrak{G}_1$ commute with those of $\mathfrak{G}_2$, $T(g_1)$ belongs to R_2' and $T(g_2)$ to R_1'. Hence, it follows that $R_1 \subset R_2'$, $R_2 \subset R_1'$. But the rings R_1' and R_2' intersect only in operators that are multiples of unit operators. (For every operator belonging both to R_1' and R_2' commutes with all the operators $T(g)$ of an irreducible representation of $\mathfrak{G}$, consequently it is a multiple of the unit operator.) Since $R_1 \subset R_2'$, we conclude that R_1 and R_1' intersect only in operators that are multiples of the unit operator.

So we have shown that the ring R_1 of operators generated by the operators $T(g_1)$, $g_1 \in \mathfrak{G}_1$, is a factor. By hypothesis, this factor is of type I. This means that H is the tensor product of H_1 and H_2: $H = H_1 \otimes H_2$, that R_1 consists of all operators of the form $A \otimes 1$, and R_1' of all operators of the form $1 \otimes B$.

Hence, it is clear that the representation operators are of the form

$$T(g_1 g_2) = T_1(g_1) \otimes T_2(g_2).$$

LEMMA 2. *Let G be a topological group and let $T(g)$ be a unitary representation of it in a Hilbert space H. If $\mathfrak{G}$ contains a sequence of compact subgroups V_n, $n = 1, 2, \ldots$, converging to the trivial group, then for*

† For the definition of a group of type I see the Appendix to Chapter 2.

sufficiently large n there is in H a vector f invariant under all the operators $T(v_n)$, $v_n \in V_n$.

Proof. We introduce the operator

$$P_n = \int T(v_n)\, dv_n, \tag{1}$$

where the integration is taken with respect to the invariant measure dv_n on V_n, normalized by the condition

$$\int dv_n = 1.$$

We show that the sequence of operators P_n strongly converges to the unit operator. For by definition of the topology on $\mathfrak{G}$, the sequence

$$v_1, \ldots, v_n, \ldots,$$

where $v_n \in V_n$, converges, in fact uniformly with respect to v_n, to the unit element of $\mathfrak{G}$. Hence, it follows that the sequence of operators

$$T(v_1), \ldots, T(v_n), \ldots$$

strongly converges, uniformly with respect to v_n, to the unit operator. Hence, for every vector $f \in H$ and every $\varepsilon > 0$ we can find an N such that for $n \geq N$ we have

$$\| T(v_n)f - f \| < \varepsilon, \tag{2}$$

no matter what $v_n \in V_n$ we take.

From (2) it follows immediately that

$$\|P_n f - f \| < \varepsilon,$$

that is, the sequence of operators P_n strongly converges to the unit operator.

Since the P_n strongly converge to the unit operator, we can find an n such that $P_n \neq 0$. We show that then H contains a vector invariant under the $T(v_n)$. For let $\varphi \neq 0$ be an arbitrary vector of the form $\varphi = P_n f$. Then we have $T(v_n)\varphi = T(v_n)P_n f$. But from the definition of P_n it follows immediately that $T(v_n)P_n = P_n$ for every $v_n \in V_n$. Consequently, $T(v_n)\varphi = \varphi$, that is, φ is the required vector invariant under the $T(v_n)$. And so Lemma 2 is proved.

Now we proceed to the proof of the theorem.

Let $T(g)$ be a given irreducible unitary representation of the group of adeles G_A, acting in a Hilbert space H. With this representation we associate an irreducible representation $T_p(g_p)$ of each of the groups G_p, $p = \infty, 2, 3, \ldots$.

For this purpose we observe that G_A splits into the direct product

$$G_A = G_p \times G'_p$$

of G_p and the subgroup G_p' consisting of all adeles $g = (g_\infty, g_2, \ldots)$ in which $g_p = 1$. By Lemma 1 the representation $T(g)$ is the tensor product of an irreducible representation $T_p(g_p)$ of G_p and an irreducible representation of G_p'.

Thus, with the original representation $T(g)$ of G_A we have associated irreducible representations $T_p(g_p)$ of the subgroups G_p, $p = \infty, 2, 3, \ldots$.

Now we show these representations are not altogether arbitrary. In fact, they satisfy the following condition: for sufficiently large p the representation space H_p of $T_p(g_p)$ contains a vector f_p invariant under the operators $T_p(u_p)$ where u_p ranges over the compact subgroup U_p consisting of all integral matrices on G_p.

We denote by V_p the subgroup of adeles of the form

$$v_p = (1, \ldots, 1, u_p, \ldots u_q, \ldots),$$

where $u_q \in U_q$ for $q \geq p$. The sequence of subgroups V_p converges to the trivial group. Therefore, by Lemma 2 there exists a $p = p_0$ for which H contains a vector f invariant under the operators $T(v_{p_0})$, $v_{p_0} \in V_{p_0}$. Let $p \geq p_0$. We decompose H into the tensor product

$$H = H_p \otimes H_p'$$

of the space H_p in which the irreducible representation $T_p(g_p)$ of G_p acts and the space H_p' in which G_p' acts. Then the invariant vector f can be written uniquely in the form of a sum

$$f = \sum_{i=1}^{\infty} f_{p,i} \otimes \psi_i,$$

where $f_{p,i}$ are vectors from $\tilde{H}_p$, and the ψ_i range over a fixed orthogonal basis of H_p'. Since $U_p \subset V_{p_0}$, we have

$$T(u_p)f = f$$

for every $u_p \in U_p$. But

$$T(u_p)f = \sum_{i=1}^{\infty} (T_p(u_p)f_{p,i}) \otimes \psi_i,$$

and therefore,

$$T_p(u_p)f_{p,i} = f_{p,i},$$

that is, each of the vector $f_{p,i} \in H_p$ is invariant under the operators $T_p(u_p)$. This proves the proposition.

Irreducible unitary representations $T_p(g_p)$ of G_p having at least one vector invariant under the operators $T_p(u_p)$, where u_p ranges over the subgroup U_p of integral matrices are called *representations of class I.*

Thus, with every irreducible unitary representation $T(g)$ of the group of adeles G_A we have associated a sequence of irreducible unitary representations $T_p(g_p)$ of each subgroup $G_p, p = \infty, 2, 3, \ldots$.

All the representations $T_p(g_p)$, except possibly a finite number, are of class I. Our task is now to give a description of $T(g)$ in terms of the representations $T_p(g_p)$.

Again we consider the subgroup V_p of adeles of the form $v_p = (1, \ldots, 1, u_p, \ldots, u_q, \ldots)$, where $u_q \in U_q$ when $q \geq p$.

As we know already, there exists a $p = p_0$ such that H contains a vector f invariant under the operators $T(v_{p_0})$, $v_{p_0} \in V_{p_0}$; hence, all the representations $T_p(g_p)$ are of class I for $p \geq p_0$.

We choose an orthonormal basis $f_{p,1}, \ldots, f_{p,n}, \ldots$ in the representations space H_p of $T_p(g_p)$; we assume that for $p \geq p_0$ the vector $f_{p,1}$ is invariant under the operators $T_p(u_p)$, $u_p \in U_p$.

By Lemma 1, H can be represented for every p as a tensor product

$$H = \left(\bigotimes_{q<p} H_q\right) \otimes H_p'', \tag{3}$$

where H_p'' is the space in which the irreducible unitary representation $T_p''(g_p'')$ of the group G_p'' of adeles of the form

$$g = (1, \ldots, 1, g_p, \ldots),$$

acts.

It is easy to verify that among the vectors in H that are invariant under the operators $T(v_{p_0})$ there is a vector of the following form:

$$f = \prod_{q<p_0} f_{q,1} \cdot f',$$

where f' is a vector in H_{p_0}'' invariant under the operators $T_{p_0}''(v_{p_0})$. We assume that $\|f'\| = 1$.

Next, it is easy to check that for every $p \geq p_0$ the vector f' has, by virtue of (3), the following form

$$f' = \prod_{p \leq q < p} f_{q,1} \cdot f_p'', \tag{4}$$

where $f_p'' \in H_p''$. By (4), f_p'' can be written in the following form

$$f_p'' = \prod_{q>p} f_{q,1}.$$

In H we consider all vectors of the form

$$\prod_{q<p} f_{q,i_q} \cdot f_p'' = \prod_{q<p} f_{q,i_q} \cdot \prod_{q \geq p} f_{q,1}, \tag{5}$$

where $p \geq p_0$; $i_q > 1$ in the last factor f_{q,i_q}; the vector f_p'' is defined by (4). Obviously, these vectors form an orthonormal system in H.

The representation operator $T(g)$, $g = (g, g_2, \ldots, g_p, \ldots)$ acts on these vectors in the following way:

$$\begin{aligned} T(g) \prod_{q<p} f_{q,i_q} \cdot f_p'' &= \prod_{q<p} (T_q(g_q) f_{q,i_q})(T_p''(g_p'') f_p'') \\ &= \prod_{q<p} (T_q(g_q) f_{q,i_q}) \prod_{p \leq q < p'} (T_q(g_q) f_{q,i_q})(T_{p'}''(g_{p'}'') f_{p'}''). \end{aligned}$$

Here $g_p'' = (1, \ldots, 1, g_{p'}, \ldots, g_q, \ldots)$.

Hence, it is easy to see that the vector $T(g)\prod_{q<p} f_{q,i_q}\cdot f''_p$ is again a linear combination of vectors of the form (5). This follows immediately from the fact that for every $g\in G_A$ there exists a $p'\geq p_0$ such that $g''_{p'}\in V_{p'}$; consequently, we have

$$T''_{p'}(g''_{p'})f''_{p'}=f''_{p'}. \tag{6}$$

Thus, the subspace H' spanned by the vectors (5) is invariant under the operators $T(g)$. Since $T(g)$ is irreducible, we have $H'=H$, that is, the vectors (5) form a *complete* orthonormal system in H.

Formula (6) is what we seek. It expresses the given representation $T(g)$ of the group of adeles G_A in terms of the representation $T_p(g_p)$ of the groups G_p.

Remark. It would be interesting to find out if, for a given semisimple algebraic group G, only finitely many of the groups G_p can have irreducible unitary representations containing more than one linearly independent vector invariant under the subgroups U_p.

4. Criteria for the Existence of a Single Linearly Independent Invariant Vector. Here we give a sufficient condition for an irreducible representation of a topological group G to contain not more than one vector invariant under a compact subgroup of G.

Let G be locally compact topological group, with a measure dg, invariant both under left and right translations† *(that is, $d(g_1gg_2)=dg$ for any g_1, $g_2\in G$). Let U be a compact subgroup of G. We assume that G admits a map*

$$\sigma:\ g\rightarrow g^\sigma,$$

with the following properties:

1. *σ^2 is the identity map.*

2. *σ is an anti-isomorphism, effecting a one-to-one map of G onto itself and satisfying for arbitrary g_1, $g_2\in G$ the following condition:*

$$(g_1g_2)^\sigma=g_2^\sigma g_1^\sigma.$$

3. *For every $g\in G$ there exist u_1, $u_2\in U$ such that*

$$g^\sigma=u_1gu_2.$$

As a consequence of these properties we have:

4. *$U^\sigma=U$, that is, σ maps the compact subgroup U onto itself;*

5. *$d(g^\sigma)=dg$, that is, σ preserves the measure.*

We show that every irreducible unitary representation G has not more than one linearly independent vector invariant under U.

† Observe that from the two-sided invariance of dg, it follows that this measure is also inverse-invariant, that is, $dg^{-1}=dg$.

Proof. We consider the collection R_0 of continuous finite functions $\sigma(g)$ of G that satisfy the following condition:

$$\varphi(u_1 g u_2) = \varphi(g) \quad \text{for arbitrary } u_1, u_2 \in U. \tag{1}$$

In R_0 we introduce, in a natural way, an operation of addition and we define multiplication of two functions from R_0 as the convolution

$$\varphi_1 * \varphi_2(g) = \int \varphi_1(g_1)\, \varphi_2(g_1^{-1} g)\, dg_1.$$

It is easy to check directly that the convolution of two functions in R_0 is again a function in R_0. Thus, R_0 is a ring.

We show that *the ring R_0 is commutative.*

For this purpose we observe that by 3 a function $\varphi \in R_0$ satisfies the relation

$$\psi(g^\sigma) = \psi(g).$$

Consequently,

$$\begin{aligned} \psi_1 * \psi_2(g) &= \int \psi_1(g_1)\, \psi_2(g_1^{-1} g)\, dg_1 \\ &= \int \psi_1(g_1^\sigma)\, \psi_2(g^\sigma (g_1^\sigma)^{-1})\, dg_1 = \int \psi_1(g_1)\, \psi(g^\sigma g_1^{-1})\, dg_1. \end{aligned}$$

The last integral is a function in R_0, consequently its value does not change when g^σ is replaced by g. In other words, we have

$$\begin{aligned} \psi_1 * \psi_2(g) &= \int \psi_1(g_1)\, \psi_2(g g_1^{-1})\, dg_1 = \int \psi_1(g_1^{-1})\, \psi_2(g g_1)\, dg_1 \\ &= \int \psi_2(g_1)\, \psi_1(g_1^{-1} g)\, dg_1. \end{aligned}$$

So we have shown that $\psi_1 * \psi_2 = \psi_2 * \psi_1$.

Now let $T(g)$ be a unitary (but not necessarily irreducible) representation of G. Then we can associate with every element $\varphi \in R_0$ the operator

$$T_\varphi = \int \varphi(g)\, T(g)\, dg.$$

An immediate verification shows that

$$T_{\lambda_1 \varphi_1 + \lambda_2 \varphi_2} = \lambda_1 T_{\varphi_1} + \lambda_2 T_{\varphi_2},$$
$$T_{\varphi_1 * \varphi_2} = T_{\varphi_1} T_{\varphi_2}$$

for arbitrary $\varphi_1, \varphi_2 \in R_0$ and any complex numbers λ_1, λ_2; consequently, the correspondence

$$\varphi \to T_\varphi$$

is a representation of R_0.

Thus, *to every unitary representation $T(g)$ of G there corresponds a representation of the commutative ring R_0.*

We show that *the operators T_φ, $\varphi \in R_0$, have the following property:*

$$T(u_1)T_\varphi T(u_2) = T_\varphi \tag{2}$$

for arbitrary $u_1, u_2 \in U$.

For by definition we have

$$T_\varphi = \int \varphi(g)T(g)\,dg;$$

consequently,

$$T(u_1)T_\varphi T(u_2) = \int \varphi(g)T(u_1gu_2)\,dg$$

$$= \int \varphi(u_1^{-1}gu_2^{-1})T(g)\,dg = \int \varphi(g)T(g)\,dg = T_\varphi.$$

We denote by $\mathfrak{M}$ the subspace of those vectors f of the representation space of $T(g)$ for which

$$T(u)f = f \quad \text{for every } u \in U.$$

From (2) it follows that *the operators T_φ, $\varphi \in R_0$, carry the representations space H of $T(g)$ into $\mathfrak{M}$. In particular, $\mathfrak{M}$ is invariant under the operators T_φ, $\varphi \in R_0$.*

For by (2) we have

$$T(u)(T_\varphi f) = T_\varphi f$$

for every $f \in H$ and every $u \in U$.

We have to show that if $T(g)$ is an irreducible representation, then the dimension of $\mathfrak{M}$ is either 0 or 1.

Suppose the contrary: the dimension of $\mathfrak{M}$ is greater than 1. Since R_0 is a commutative ring, the operators T_φ, $\varphi \in R_0$, form a commutative system, because $\mathfrak{M}$ necessarily contains a proper invariant subspace $\mathfrak{M} \neq 0$. We fix a vector $f' \neq 0$ in $\mathfrak{M}'$ and a vector $f \in \mathfrak{M}$ orthogonal to $\mathfrak{M}'$.

We consider the set $\bar{H}'$ of vectors of the form $T_\psi f'$, where ψ ranges over all finite continuous functions on G, and

$$T_\psi = \int \psi(g)T(g)\,dg.$$

Let $\bar{H}'$ be the closure of this set in H.

It is not hard to see that $\bar{H}'$ is a linear subspace of H, invariant under the operators of $T(g)$, and $\bar{H}' \neq 0$.

We show that $\bar{H}'$ is orthogonal to f. Hence, it will follow that $\bar{H}'$ is a proper invariant subspace of H, and this contradicts the irreducibility of H.

Since $T(u)f' = f'$ and $T(u)f = f$ for every $u \in U$, we have

$$(T_\varphi f', f) = (T(u_1) T_\varphi T(u_2) f', f)$$

for arbitrary u_1 and u_2. We integrate both sides of this equation with respect to u_1 and u_2 and assume that the measure of the compact subgroup U is equal to 1. We find that

$$(T_\varphi f', f) = (\hat{T} f', f),$$

where

$$\hat{T} = \int T(u_1) T_\varphi T(u_2)\, du_1\, du_2 = \int \varphi(g) T(u_1 g u_2)\, dg\, du_1\, du_2$$

$$= \int \varphi(u_1^{-1} g u_2^{-1}) T(g)\, dg\, du_1\, du_2.$$

Setting

$$\hat{\varphi}(g) = \int \varphi(u_1 g u_2)\, du_1\, du_2,$$

we have

$$\hat{T} = T_{\hat{\varphi}}.$$

Obviously $\hat{\varphi}$ satisfies the relation $\hat{\varphi}(u_1 g u_2) = \hat{\varphi}(g)$ for arbitrary u_1, $u_2 \in U$, and so $\hat{\varphi} \in R_0$. But then $\hat{T}f' \in \mathfrak{M}'$, and consequently $(\hat{T}f', f) = 0$.

Thus, we have shown that $(T_\varphi f', f) = 0$, for every finite function φ. Consequently, f is orthogonal to $\bar{H}'$, and the theorem is proved.

5. Second Theorem on Tensor Products. Here we establish another criterion for an irreducible unitary representation $T(g)$ of G_A to be a tensor product of representations of the groups G_p.

By analogy with the group of adeles A, we introduce the concept of a Schwartz-Bruhat function on G_A.

We denote by S_∞ the space of infinitely differentiable quickly decreasing functions on G_∞, and by S_p the space of finite piecewise constant functions on G_p, $p = 2, 3, \ldots$.

We consider the functions on G_A that are representable as an infinite product

$$\varphi(g) = \varphi_\infty(g_\infty)\varphi_2(g_2) \cdots \varphi_p(g_p) \cdots, \tag{1}$$

where the factors $\varphi_p(g_p)$ satisfy the following conditions:

1. $\varphi_\infty \in S_\infty$; $\varphi_p \in S_p$, $p = 2, 3, \ldots$
2. For all p except a finite number $\varphi_p(g_p)$ is concentrated on the subgroup U_p of integral matrices and is identically equal to 1 on this subgroup.

We define a *Schwartz-Bruhat function* on G_A as a function $\varphi(g)$ that is representable as a finite linear combination of functions of the form (1).

Now let $T(g)$ be an irreducible unitary representation of G_A. We set

$$T_\varphi = \int \varphi(g) T(g)\, dg.$$

Then we have the following proposition.

If for every Schwartz-Bruhat function φ the operator T_φ is completely continuous and has a trace, then $T(g)$ is a tensor product of irreducible unitary representations $T_p(g_p)$ of the groups G_p. Furthermore, each representation space $T_p(g_p)$, beginning with a sufficiently large p, has one and only one linearly independent vector invariant under the subgroup U_p of integral matrices.

Proof. The first part of the proposition, namely, that $T(g)$ is a tensor product of the representations $T_p(g_p)$ and that each representation space $T_p(g_p)$ beginning with sufficiently large p contains at least one vector invariant under U_p, is proved almost word for word as Theorem 1 in **§ 3.3**. The only difference is that the proof is based not on Lemma 1 proved on page 275, but on the following lemma proved in Chapter 2, Appendix, page 223.

We assume that an irreducible representation $T(g) \equiv T(g_1, g_2)$ of the group $G = G_1 \times G_2$ has the following property.

There exists a function φ, summable on G, of the form

$$\varphi(g_1, g_2) = \varphi_1(g_1)\varphi_2(g_2)$$

for which the operator

$$T_\varphi = \int \varphi(g_1, g_2) T(g_1, g_2)\, dg_1\, dg_2$$

is nonzero and completely continuous.

Then the representation $T(g)$ is a tensor product of irreducible representations of G_1 and G_2.

We show that each representation space $T_p(g_p)$, beginning with a sufficiently large p, contains only one linearly independent vector invariant under U_p.

By what we have proved, there is a $p = p_0$ such that for $p \geq p_0$ the representation space of $T_p(g_p)$ contains at least one vector invariant under U_p. We consider a Schwartz-Bruhat function of the form

$$\varphi(g) = \varphi_\infty(g_\infty)\varphi_2(g_2) \cdots \varphi_p(g_p) \cdots,$$

where $\varphi_p(g_p)$ for $p \geq p_0$ is the characteristic function of the set $U_p \subset G_p$.

Then we have

$$T_\varphi = T_{\varphi_\infty} \otimes T_{\varphi_2} \otimes \ldots \otimes T_{\varphi_p} \otimes \ldots$$

and

where
$$T_{\varphi_p} = \int_{G_p} \varphi_p(g_p) T_p(g_p)\, dg_p$$
is an operator in the representation space of $T_p(g_p)$, and Tr $T_{\varphi p}$ is the trace of this operator, $p = \infty, 2, 3, \ldots$.

For $p = \infty$ and for $p < p_0$ we assume the function $\varphi_p(g_p)$ to be chosen so that Tr $T_{\varphi_p} \neq 0$.

Now we observe that for $p \geq p_0$ the operator T_{φ_p} has the following form:
$$T_{\varphi_p} = \int_{U_p} T_p(g_p)\, dg_p.$$
Hence, it is a projection operator onto the subspace of vectors invariant under u_p. Therefore, Tr $T_{\varphi_p} = n_p$ for $p \geq p_0$, where n_p is the maximal number of linearly independent vectors in the representation space that are invariant under U_p.

We assume that $n_p > 1$ for an infinite set of p. Then the product (2) diverges, but this contradicts the hypothesis on the existence of a trace of T_φ. So the proposition is proved.

§ 4. THE ADELE GROUP OF THE GROUP OF UNIMODULAR MATRICES OF ORDER 2

1. Statement of the Problem and Summary of the Results. Let G be the group of unimodular matrices of order 2 over the field of rational numbers Q, and $\mathfrak{G} = G_A$ the group of adeles associated with it. We denote by $\Gamma = G_Q$ the subgroup of principal adeles of G_A. As was proved in **§ 2**, Γ is a discrete subgroup of G_A. The present section deals with the problem of decomposing the representation of G_A generated by the homogeneous space $X = \Gamma \setminus G_A$ into irreducible representations.

The relationship of this problem to the corresponding one discussed in Chapter 1, where $\mathfrak{G}$ is the group of real matrices of order 2 and Γ a discrete subgroup of it, will be clarified in **§ 5,** Appendix II. The methods to be applied here are similar to those of Chapter 1. We now state the main results of this section. As a preliminary we introduce the concept of a horospherical subgroup and a horosphere (see Chap. 1, **§ 6**). A subgroup of adeles of the form
$$z = (z_\infty, z_2, \ldots, z_p, \ldots),$$
where $z_p = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ ($*$ denotes an arbitrary p-adic number), and any subgroup conjugate to it are called horospherical subgroups.

Horospheres in the homogeneous space X are defined as orbits

of a horospherical subgroup. We are mainly interested in compact horospheres.

It is not hard to check that the transformation $x \to xg$, $g \in G_A$, carries a horosphere into a horosphere, and a compact horosphere into a compact horosphere. In **§ 4.3** we shall show that in the space $X = \Gamma \setminus G_A$ the set of all compact horospheres is transitive, that is, for every pair of compact horospheres there exists a transformation carrying one into the other. In other words, the space Ω of all compact horospheres of X is homogeneous. As we shall show in **§ 4.7**, this space has an invariant measure.

We introduce the horospherical map of the space $L_2(X)$, that associates with every function $f(x) \in L_2(X)$ a function $\varphi(\omega)$ on Ω: $\varphi(\omega)$ is defined as the integral† of $f(x)$ over the horosphere ω.

It can be shown (see **§ 4.5**) that the kernel H^0 of the horospherical map is a closed invariant subspace of $L_2(X)$.

Let H' be the image of $L_2(X)$ under the horospherical map. We endow H' with the structure of a Hilbert space by setting $H' \cong L_2(X)/H^0$. Then $L_2(X)$ is isomorphic to the direct sum

$$L_2(X) \cong H^0 \oplus H',$$

so that the investigation of the spectrum of $L_2(X)$ reduces to that of the spectra of H^0 and H'.

In **§ 4.6** it will be shown that the space H^0 has a countable spectrum of finite multiplicity, that is, the representation of G_A in H^0 splits into the direct sum of a countable number of irreducible representations, and each of them occurs with finite multiplicity.

The principal task of this section is the study of the spectrum of the space H'. The main result consists in the following (see **§ 4.17**). *The space H' is the direct sum*

$$H' = C \oplus H''$$

of the subspace of constants C and a subspace $H'' \subset L_2(\Omega)$. Here H'' contains all irreducible representations of G_A that occur in $L_2(\Omega)$, and each of them with multiplicity 1 (*in contrast to $L_2(\Omega)$ itself*, where they occur with multiplicity 2).

We mention that the classification of the spectrum of $L_2(\Omega)$ is an elementary problem (it is solved in **§ 4.9**).

Now we indicate briefly the way in which this result is achieved. Two operators play the main role in the proof:

$$B\colon\ S(\Omega) \to C(\Omega)$$

and

$$M\colon\ \varphi(\Omega) \to H',$$

† The measures on all compact horospheres are normalized so that each of them has measure 1.

where $S(\Omega)$, $\varphi(\Omega)$, and $C(\Omega)$, respectively, are the spaces of fast declining, finite, and continuous functions on Ω.

The operator B is defined as follows. In the space Ω there is given a set of closed surfaces (horospheres), forming a homogeneous space isomorphic to Ω. If $\psi(y) \in S(\Omega)$, and ω is an arbitrary horosphere, we define $(B\psi)(\omega)$ as the integral of $\psi(y)$ over the horosphere ω. Thus, $B\psi$ is a function on the set of horospheres in Ω. Since this set is isomorphic to Ω, $B\psi$ can again be regarded as a function on Ω.

MAIN THEOREM ON THE OPERATOR B (Theorem 5). *There exists a subset ψ_c of functions on Ω having the following properties:*

1. *ψ_c is everywhere dense in $L_2(\Omega)$.*
2. *$B\psi \in L_2(\Omega)$ for every $\psi \in \Psi_c$.*
3. *The operator B can be extended from the subset ψ_c to a unitary operator $\bar{B}$ on the whole space $L_2(\Omega)$. Here $\bar{B}$ satisfies the relation: $\bar{B}^2 = E$, where E is the unit operator.*

An effective construction of this family Ψ_c and a proof of the theorem will be given in **§ 4.12** and **§ 4.13.**

On the basis of this theorem it is not hard to show that $B_\psi = \bar{B}_\psi$ for every finite function ψ satisfying the condition: $B\psi \in L_2(\Omega)$.

The second operator M is defined in the following way. Every function $\psi \in \Phi(\Omega)$ gives rise to a continuous linear functional (ψ, φ) in the space H' defined by the formula

$$(\psi, \varphi) = \int_\Omega \varphi(y)\overline{\psi(y)}\, dy, \qquad \varphi \in H'.$$

Consequently, by the theorem of Riesz, there exists a function $\hat{\psi} \in H'$ such that $(\psi, \varphi) = [\varphi, \hat{\psi}]$ for every $\varphi \in H'$; the brackets refer to the scalar product in H'. We set, by definition, $M\psi = \hat{\psi}$.

It can be established that a simple relationship holds between the operators B and M:

$$M = E + B,$$

where E is the unit operator.

MAIN THEOREM ON THE OPERATOR M (Theorem 9). *There exists a set $\mathfrak{M}$ of finite functions defined on Ω and having the following properties:*

1. *$\mathfrak{M}$ is everywhere dense in $L_2(\Omega)$.*
2. *$M \in L_2(\Omega)$ for every $\psi \in \mathfrak{M}$.*
3. *The functions $M\psi$, $\psi \in \mathfrak{M}$, belong to H'' (H'' is the orthogonal complement in H' to the subspace of constants) and form in H'' an everywhere dense subset.*

An effective construction of this family $\mathfrak{M}$ and a proof of the theorem are given in **§ 4.16.** We mention that the proof of the

main theorems on the operators B and M are based on a property of the Dirichlet L-functions.

On the basis of the theorems on the operators B and M it is easy to prove the following proposition:

The operator M can be extended from $\mathfrak{M}$ to a bounded self-adjoint operator $\bar{M} = E + \bar{B}$ defined on $L_2(\Omega)$ and satisfying the relation $\bar{M}^2 = 2\bar{M}$; the image of $L_2(\Omega)$ under the map $\bar{M}$ coincides with H''.

Hence, it follows that $H'' \subset L_2(\Omega)$.

The proof that H'' has a spectrum of simple multiplicity is based on a property of the operator $\bar{B}$, which will be established in **§ 4.13.**

2. The Structure of the Space X. In this subsection we show that $X = \Gamma \setminus G_A$ *is a fiber bundle whose base is the space* $G_Z \setminus G_\infty$ *of cosets of the subgroup of integral matrices* G_Z *in* G_∞, *and whose fiber is the group* $U = \prod_p U_p$, *where* U_p *denotes the set of all integral p-adic matrices of order* 2 *with determinant* 1.

First we prove the following lemma.

Let g be a matrix of order 2 with elements from the field of p-adic numbers and with determinant 1. Then there exists a matrix γ with rational elements such that:

1. $\gamma g \in U_p$,
2. $\gamma \in U_q$ *for every* $q \neq p$.

For let $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We denote by a_n, b_n, c_n, d_n, rational numbers that are congruent, respectively, to a, b, c, d modulo p^n, and whose denominators are powers of p. We set

$$\gamma_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

It is not hard to verify that a_n, b_n, c_n, d_n can always be chosen so that the matrix γ_n is unimodular (see the analogous proposition on p. 112). Obviously, γ_n satisfies for all n condition 2 of the lemma, and for sufficiently large n also condition 1. So the proof of the lemma is complete.

Let $g = (g_\infty, g_2, \dots, g_p, \dots)$ be an arbitrary element of G_A. Since there are only finitely many $g_p \notin U_p$, it follows from the lemma that there exists a $\gamma \in \Gamma$ for which $\gamma g = (h_\infty, h_2, \dots, h_p, \dots)$ where $h_p \in U_p$ for $p = 2, 3, \dots$.

In other words, every coset Γg contains an element $h \in \Gamma g$ such that $h_p \in U_p$ for all p except $p = \infty$.

Let us find out under what conditions two elements h and h'

of this form belong to one and the same coset Γg of Γ in G. Obviously, if h, $h' \in \Gamma g$, then $h'_\infty = \gamma h_\infty$, where γ is some integral matrix, since for $p = 2, 3, \ldots$ both h_p and $h'_p = \gamma h_p$ are integral p-adic matrices. In other words, the first co-ordinates h_∞ of these elements belong to the same coset $G_Z h_\infty$ of the subgroup G_Z of all integral matrices in G_∞.

So we have established a map

$$X = \Gamma \setminus G_A \xrightarrow{\tau} G_Z \setminus G_\infty$$

of X into the space of cosets of G_Z in G_∞. It is not hard to verify that this map τ is continuous.

Let us find the complete inverse images of the elements in $G_Z \setminus G_\infty$ under τ. Clearly, $\tau(x) = \tau(x')$ if and only if $x' = xu$, where u belongs to the subgroup U of elements of the form

$$(1, u_2, \ldots, u_p, \ldots), \qquad u_p \in U_p. \tag{1}$$

On the other hand, it is easy to check that $xu = x$ only for $u = 1$. Hence, the complete inverse image $\tau^{-1}(y)$ any element $y \in G_Z \setminus G_\infty$ is isomorphic to U.

So we have established that X is a fiber bundle with the base $G_Z \setminus G_\infty$ and fiber isomorphic to U.

The verification that this fibering locally has the structure of a direct product is trivial. Note that the fiber space itself is, of course, not a direct product. Its monodromy group is isomorphic to G_Z, as is easy to verify.

Without proof we mention that the analogous result on the structure of the homogeneous space $X = G_Q \setminus G_A$ is also valid for every linear algebraic group G.

We recall that the homogeneous space $G_Z \setminus G_\infty$ of G_∞ has finite volume (see Chap. 1, App.). Since U is a compact group, it now follows immediately from our result above that *the volume of the homogeneous space X is finite.*

3. Description of the Space Ω of all Compact Horospheres of X.

We recall that horospheres in X are defined as orbits of a horospherical subgroup, that is, a subgroup $gZ_A g^{-1}$, where Z_A is a group of adeles of the form

$$z = (z_\infty, z_2, \ldots, z_p, \ldots),$$

for which $z_p = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ ($*$ denotes any p-adic number), and g is an arbitrary fixed element G_A.

We wish to describe the space Ω.

Let D_Q be the subgroup of Γ consisting of the adeles of form

$$(\delta, \delta, \ldots, \delta, \ldots),$$

where

$$\delta = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

and λ ranges over all nonzero rational numbers. Let us investigate the subgroup $D_Q Z_A$ generated by D_Q and Z_A.

In this subsection we prove the following theorem.

The space Ω of compact horospheres is homogeneous with stability group $D_Q Z_A$.

We divide the proof into several steps. First, we consider a horosphere $x_z = x_0 z$, where x_0 is the point of $X = \Gamma \setminus G_A$ corresponding to the unit class, and z ranges over Z_A. We show that *the horosphere $x_z = x_0 z$ is compact.*

Since the stability group of x_0 is Γ, the set of all horospheres $x_z = x_0 z$ is obviously homeomorphic to the space $Z_Q \setminus Z_A$. Thus we show that $Z_Q \setminus Z_A$ is a compact space.

Observe that Z_A is isomorphic to the group A of adeles

$$a = (a_\infty, a_2, \ldots, a_p, \ldots),$$

where a_∞ ranges over the additive group of real numbers, a_p over the additive group of p-adic numbers, and all a_p, from some p onward, are p-adic integers. So we have

$$Z_Q \setminus Z_A \cong Q \setminus A,$$

where Q is the subgroup of principal adeles of A. The compactness of the space $Q \setminus A$ was proved in **§ 1.4.**

Hence, we have proved the compactness of horospheres $x_z = x_0 z$.

Now we show that *every compact horosphere in X can be obtained by a translation of the horosphere $x_z = x_0 z$, that is, the set of compact horospheres is transitive.*

Since a translation of a compact horosphere is again compact, it suffices to consider only horospheres of the form $x_z = x_0 g z g^{-1}$. Thus, let $x_z = x_0 g z g^{-1}$ be a compact horosphere. Then the intersection $\Gamma \cap g Z_A g^{-1}$ of the group $g Z_A g^{-1}$ with the stability group Γ of x_0 is not trivial. Hence, there exists a principal adele

$$\gamma = (\gamma, \gamma, \ldots, \gamma, \ldots),$$

$\gamma \neq e$, that is representable in the form

$$\gamma = g z g^{-1}, \tag{1}$$

where z is some element of Z_A.

The decomposition $\gamma = g z g^{-1}$ shows that the eigenvalues of the rational matrix γ are equal to 1. But then there is a rational matrix

γ_0 such that $\gamma_0^{-1}\gamma\gamma_0$ is a matrix of the form $\begin{pmatrix} 1 & 0 \\ z_0 & 1 \end{pmatrix}$. Denoting by the same letter γ_0 the principal adele $\gamma_0 = (\gamma_0, \ldots, \gamma_0, \ldots)$, we have

$$\gamma_0^{-1}\gamma\gamma_0 = z_0 \in Z_A,$$

Thus,

$$\gamma = \gamma_0 z_0 \gamma_0^{-1},$$

where $\gamma_0 \in \Gamma$, $z_0 \in Z \cap \Gamma = Z_Q$, and therefore (1) can be rewritten in the form

$$\gamma_0 z_0 \gamma_0^{-1} = gzg^{-1}. \tag{2}$$

But from (2) it follows that the adele $\gamma_0^{-1}g$ is of the form

$$\gamma_0^{-1}g = k = (k_\infty, k_2, \ldots, k_p, \ldots), \tag{3}$$

where the k_p are triangular matrices of the form

$$k_p = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \tag{3'}$$

So we see that if the horosphere $x_z = x_0 gzg^{-1}$ is compact, then the adele g has the following form:

$$g = \gamma_0 k,$$

where $\gamma_0 \in \Gamma$, and k is an adele of the form (3). But since obviously

$$kZ_A k^{-1} = Z_A,$$

the equation of the horosphere $x_z = x_0 gzg^{-1}$ can be written in the form $x_z = x_0\gamma_0 z\gamma_0^{-1}$ or, because $x_0\gamma_0 = x_0$, in the form $x_z = x_0 z\gamma_0^{-1}$. So we have proved that every compact horosphere in X can be obtained by a translation of the horosphere $x_0 z$.

Hence, every compact horosphere in the space $X = \Gamma \setminus G_A$ is given by the equation

$$x_z = x_0 zg,$$

where x_0 is the point of X corresponding to the unit class. Consequently, the set Ω of compact horospheres in X is a homogeneous space relative to the group G_A.

Finally, we show that the stability group of the horosphere $x_z = x_0 z$ is $D_Q Z_A$. Suppose that g carries the horosphere x_z into itself. Then we can represent x_0 in the form $x_0 = x_0 zg$. Consequently, there exists $\gamma \in \Gamma$, such that $1 = \gamma zg$, where 1 is the unit element of G_A. Hence we have: $g = z^{-1}\gamma^{-1}$, where $z^{-1} \in Z_A$, $\gamma^{-1} \in \Gamma$.

Without loss of generality we may assume that z is the unit element. Then $g = \gamma_0$ is a principal adele.

From the fact that the horospheres $x_0 z$ and $x_0 z \gamma_0$ coincide, it follows that every element $z \in Z_A$ is representable in the form

$$z = \gamma z' \gamma_0, \quad \gamma \in \Gamma, \quad z' \in Z_A.$$

In this equation we go over from the adeles to their first components, and we set

$$z_\infty = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad z'_\infty = \begin{pmatrix} 1 & 0 \\ x' & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma_0^{-1} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}.$$

We find that

$$\begin{pmatrix} a_0 & b_0 \\ c_0 + a_0 x & d_0 + b_0 x \end{pmatrix} = \begin{pmatrix} a + bx' & b \\ c + dx' & c \end{pmatrix},$$

hence, in particular,

$$c = d_0 + b_0 x.$$

Since x ranges over all real numbers, and c is a rational number, it follows from this equation that $b_0 = 0$. But then it is obvious that $\gamma_0 \in D_Q Z$.

So we have shown that $D_Q Z_A$ is the stability group of the space Ω of horospheres, and the theorem is proved.

4. Cylindrical Sets. As we know, at least one compact horosphere† passes through every point $x \in X$. We consider the pairs (x, ω), where ω is a compact horosphere and $x \in \omega$. We denote by Ω_X the collection of all such pairs (x, ω), with the natural topology. There are natural continuous maps

$$\begin{tikzcd} & \Omega_X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & \Omega \end{tikzcd}$$

defined by the formulae $\pi_1(x, \omega) = x$ and $\pi_2(x, \omega) = \omega$.

Let F_0 be a subset of Ω and $F_1 = \pi_2^{-1} F_0$ its complete inverse image under π_2. We write: $F = \pi_1 F_1 = \pi_1 \circ \pi_2^{-1} F_0$. Then there are maps

$$\begin{tikzcd} & F_1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ F & & F_0 \end{tikzcd}.$$

† Note that only a discrete set of compact horospheres passes through every point $x \in X$.

We call $F_1 = \pi_2^{-1}F_0$ a *cylindrical* set if it satisfies the following conditions:

1. The map $\pi_1: F_1 \to F$ is a homeomorphism.
2. The fibering of F_1 given by the map $\pi_2: F_1 \to F_0$ is trivial.

We also call $F \subset X$ a cylindrical set, when this does not lead to confusion.

Since in the case of a cylindrical set F_1 the map $F_1 \to F$ is a homeomorphism, we have a naturally defined map

$$F \to F_0.$$

This map yields a trivial fibering of the set F:

$$F \cong (Z_Q \setminus Z_A) \times F_0,$$

whose base is the subset F_0 of compact horospheres, and whose fiber over the point $\omega \in F_0$ is the set of all $x \in \omega$.

We mention the following trivial property of cylindrical sets: if $\pi_2^{-1}F_0$ is a cylindrical set and $F_0' \subset F_0$, then $\pi_2^{-1}F_0'$ is a cylindrical set.

THEOREM 1. *For every point $\omega \in \Omega$ there exists a neighborhood F_0 such that $\pi_2^{-1}F_0$ is a cylindrical set.*

Proof. To begin with we construct a global section N in the fiber space $G_A \to Y = Z_A \setminus G_A$. We denote by $N \subset G_A$ the set of elements $g \in G_A$ of the form

$$g = (g_\infty, g_2, \ldots, g_p, \ldots),$$

where

$$g_\infty = \begin{pmatrix} a_\infty & b_\infty \\ -(a_\infty^2 + b_\infty^2)^{-1}b_\infty & (a_\infty^2 + b_\infty^2)^{-1}a_\infty \end{pmatrix},$$

$$g_p = \begin{pmatrix} a_p & b_p \\ (a_p^2 - \varepsilon_p b_p^2)^{-1}\varepsilon_p b_p & (a_p^2 - \varepsilon_p b_p^2)^{-1}a_p \end{pmatrix},$$

$$a_\infty, b_\infty \in R; \quad a_p, b_p \in Q_p; \quad \varepsilon_p \in Q_p$$

is a fixed element of norm 1 and not a square in Q_p.

An immediate verification shows that N is a section, that is, every element $g \in G_A$ has a unique representation in the form $g = zn$, $z \in Z_A$, $n \in N$; the map $(z, y) \to (zn) \to zn = g$ is a homeomorphism of the spaces $Z_A \times Y$ and G_A.

Here the invariant measure dg on G_A is expressed by the formula

$$dg = dz\, dy,$$

where dz and dy, respectively, are the invariant measures on the subgroup Z_A and in the homogeneous space Y.

Now let $\omega \in \Omega$. We have to construct a neighborhood F_0 of ω for which $\pi_2^{-1}F_0$ is a cylindrical set.

Suppose that the equation of the horosphere ω has the form

$$\omega: \quad x_z = x_0 z u_0, \qquad u_0 \in N.$$

We take a sufficiently small neighborhood $U \subset N$ of the point u_0 in N. Let F_0 be the set of all horospheres of the form

$$x_z = x_0 z u, \qquad u \in U.$$

Obviously, F_0 is an open set in Ω, and $\omega \in F_0$. We show that if the neighborhood U of the point u_0 is sufficiently small, then $\pi_2^{-1}F_0$ is a cylindrical set.

Let $F = \pi_1 \circ \pi_2^{-1}F_0$, that is, F is the set of points on the horospheres $\omega \in F_0$. Every point $x \in F$ can be represented in the form:

$$x = x_0 z u, \tag{1}$$

where $z \in Z_Q \setminus Z_A$, $u \in U$.

We show that if the domain U is sufficiently small, then the elements $z \in Z_Q \setminus Z_A$ and $u \in U$ in (1) determine the point x uniquely. For if this is not the case, then there exist sequences $z_n, z_{n'} \in Z_Q \setminus Z_A, u_n, u_{n'} \in N$ such that $(z_n, u_n) \neq (z_{n'}, u_{n'}), u_n u_n'^{-1} \to e$ and $x_0 z_n u_n = x_0 z_n' u_n'$ for every n. Let $\hat{z}_n$ and $\hat{z}_n'$ be fixed inverse images in Z_A of the elements z_n, $z_n' \in Z_Q \setminus Z_A$. Then we have $\hat{z}_n u_n = \gamma_n z_n' u_n'$ where $\gamma_n \in \Gamma$. Since $Z_Q \setminus Z_A$ is compact, we may assume that $\hat{z}_n$ and $\hat{z}_n'$ belong to a compact set; consequently, we may assume that $\hat{z}_n \to \hat{z}$, $\hat{z}_n' \to \hat{z}'$ as $n \to \infty$. But then the equation $\hat{z}_n u_n = \gamma_n \hat{z}_n' u_n'$ shows that γ_n tends to a limit belonging to $Z_Q < \Gamma$. Since Γ is discrete, γ_n belongs to Z_Q for sufficiently large n. Consequently, it follows from the equation $\hat{z}_n u_n = \gamma_n \hat{z}_n' u_n'$ that $\hat{z}_n = \gamma_n \hat{z}_n'$, $u_n = u_n'$ for sufficiently large n; hence, $(z_n, u_n) = (z_n', u_n')$. But this contradicts our assumption.

So we have shown that for a suitable choice of the domain U the elements $z \in Z_Q \setminus Z_A$ and $u \in U$ in (1) determine the point x uniquely.

Hence it follows immediately that $\pi_2^{-1}F_0$ is a cylindrical set. Indeed, the continuous map

$$\pi_1 : \pi_2^{-1}F_0 \to \pi_1 \pi_2^{-1}F_0$$

is one-to-one. Consequently, since $\pi_2^{-1}F_0$ is a compact set, π_1 is a homeomorphism. On the other hand, the decomposition (1) gives the homeomorphism

$$\pi_2^{-1}F_0 \cong (Z_Q \setminus Z_A) \cdot U \cong (Z_Q \setminus Z_A) \cdot F_0.$$

Thus, the fibering $\pi_2^{-1}F_0 \to F_0$ is trivial, and the theorem is proved.

We make a supplementary remark about the measure on a cylindrical set. Let $F \subset X$ be a cylindrical set in X, that is, F has the trivial fibering into horospheres:

$$F \cong (Z_Q \setminus Z_A) \cdot F_0, \qquad F_0 \subset \Omega. \tag{2}$$

It is easy to check that the homeomorphism (2) can always be given in such a way that the invariant measures dx, dy, dz, respectively, in the spaces $F \subset X$, $F_0 \subset \Omega$ and on the group $Z_Q \setminus Z_A$ are connected by the relations:

$$dx = dz\, dy, \qquad \text{where } x = (z, y).$$

5. The Horospherical Map. With every function $f(x) \in L_2(X)$ we associate its integral over the compact horospheres in X:

$$\varphi(g) = \int\limits_{Z_Q \setminus Z_A} f(x_0 z g)\, dz, \tag{1}$$

where x_0 is the point in X corresponding to the unit class. We call the correspondence

$$f(x) \to \varphi(g)$$

the horospherical map.

Obviously, the function $\varphi(g)$ satisfies for all $\delta \in D_Q$ and $z \in Z_A$ the following condition:

$$\varphi(\delta z g) = \varphi(g).$$

Thus, it can be regarded as a function in the space of horospheres $\Omega = D_Q Z_A \setminus G_A$, and we write $\varphi(y), y \in \Omega$, instead of $\varphi(g)$.

Let us show that *the integral* (1) *converges for all horospheres y, with the exception of a set of horospheres $y \in \Omega$ of measure* 0. *Furthermore, the function $\varphi(y)$ defined by* (1) *is summable on every compact subset $K \subset \Omega$.*

Proof. We choose an arbitrary compact cylindrical set $F \subset X$; let $F_0 \subset \Omega$ be the natural image of F in the space Ω. Then we have: $F \cong (Z_Q \setminus Z_A) \cdot F_0$ and therefore, the bounded function f on F can be regarded as a function $f(z, u)$ on $(Z_Q \setminus Z_A) \times F_0$. This function is summable on F with respect to the measure $dx = dz\, dy$. Consequently, by Fubini's theorem, the integral

$$\int\limits_{Z_Q \setminus Z_A} f(x_0 z u)\, dz \equiv \int\limits_{Z_Q \setminus Z_A} f(z, u)\, dz$$

converges for almost all $u \in F_0$ and is a summable function on F_0.

Since by Theorem 1 the sets F_0 form a covering of Ω, the integral

$$\int\limits_{Z_Q \setminus Z_A} f(x_0 z g)\, dz$$

converges for almost all horospheres $y \in \Omega$ and gives a function on Ω that is summable on every compact domain. So the proposition is proved.

THEOREM 2. *The kernel H^0 of the horospherical map is a closed subset of $L_2(X)$.*

Proof. Let F be a compact cylindrical domain in X. We denote by H_F the subspace of functions $f(x) \in L_2(X)$ for which the integrals over the horospheres into which F is fibered are zero. Then it is obvious that

$$H^0 = \bigcap H_F,$$

where the intersection is taken over the set of all compact cylindrical domains F. Thus, to prove the theorem it is sufficient to verify that every H_F is a closed subspace.

For this purpose we introduce the space $H_F^0 \subset L_2(F)$ of functions $f^* \in L_2(F)$ for which the integrals over the horospheres into which F is fibered are zero. We show that H_F^0 is closed in $L_2(F)$.

For since $F \cong (Z_Q \setminus Z_A) \cdot F_0$, $L_2(F)$ is isomorphic to the space of functions $f^*(z,y)$, $z \in Z_Q \setminus Z_A, y \in F_0 \subset \Omega$ for which

$$\|f^*\|^2 = \int_F |f^*(z,y)|^2 \, dz \, dy < \infty.$$

Here H_F^0 is isomorphic to the subspace of all functions for which

$$\int_{Z_Q \setminus Z_A} f^*(z,y) \, dz = 0 \tag{2}$$

for almost all $y \in F_0$. Since $Z_Q \setminus Z_A$ is compact, this subspace is closed† in $L_2(F)$.

Now we observe that the space $H_F \subset L_2(X)$ is the complete inverse image of the space $H_F^0 \subset L_2(F)$ under the natural map

$$L_2(X) \to L_2(F)$$

(associating with every function $f \in L_2(X)$ its restriction to F). Since this map is continuous and H_F^0 is closed, H_F is a closed subspace in $L_2(X)$, and Theorem 2 is proved.

6. Investigation of the Kernel of the Horospherical Map (Discreteness of the Spectrum). Let H^0 be the kernel of the horospherical map, that is, the subspace of functions $f(x) \in L_2(X)$ whose integrals over any compact horosphere are zero. According to **§ 4.5** this condition on $f(x)$ can be written:

$$\int_{Z_Q \setminus Z_A} f(x_0 z g) \, dz = 0 \tag{1}$$

for almost all $g \in G_A$ (x_0 denotes the point of the space $X = \Gamma \setminus G_A$

† In fact, condition (2) is equivalent to the following: $\int f^*(z,y)u(y) \, dz \, dy = 0$ for every continuous function $u(y)$ on F_0. Since $Z_Q \setminus Z_A$ is compact, we see that $(u,f^*) = \int f^*(z,y)u(y) \, dz \, dy$ is a continuous functional in $L_2(F)$. Consequently, the set of functions $f^* \in L_2(F)$ for which $(u,f^*) = 0$ is closed in $L_2(F)$.

corresponding to the unit class). In **§ 4.5** it was shown that H^0 is closed subspace of $L_2(X)$. Obviously H^0 is invariant under the representation operators of G_A:

$$T(g)f(x) = f(xg).$$

The object of this subsection is the proof of the following theorem.

THEOREM 3. *The representation of G_A in the space H^0 splits into the direct sum of at most a countable number of irreducible representations, and each representation occurs in this decomposition with finite multiplicity.*

Proof. As was shown in Chapter 1, **§ 2**, it is sufficient to prove that the operators

$$T_\varphi = \int\limits_{G_A} \varphi(g)\, T(g)\, dg$$

in the space H^0 are completely reducible for an everywhere dense set of continuous positive-definite functions $\varphi(g)$. As such a set we choose the positive-definite Schwartz-Bruhat functions† on G_A.

However, to show that the positive-definite operator T_φ is completely reducible it suffices to show that its trace is finite. Thus, to prove Theorem 3 we have to show that the trace of the operator T_φ in the space H^0 is finite for every positive-definite Schwartz-Bruhat function.

We begin by showing that X can be represented as the union of a compact and a cylindrical set. For this purpose we examine the map

$$\tau\colon \quad X \to G_Z \setminus G_\infty,$$

defined in **§ 4.2**. According to Chapter 1, **§ 6**, the space $G_Z \setminus G_\infty$ can be represented as the union of the compact set E_∞ and the cylindrical set F_∞. Let $E = \tau^{-1}E_\infty$ and $F = \tau^{-1}F_\infty$ be the inverse images of E_∞ and F_∞ under τ. Obviously, E is a compact set. We show that F is a cylindrical set, that is, it is fibered into compact horospheres, and the fibering is trivial.

For by assumption the points $x_\infty \in F_\infty$ have the form

$$x_\infty = x^0_\infty z_\infty u_\infty, \tag{2}$$

where x^0_∞ is the fixed point in $G_Z \setminus G_\infty$ corresponding to the unit class, $z_\infty \in Z_z \setminus Z_\infty$ and $u_\infty \in U_\infty$, where U_∞ is some subset of G_∞. Here the map $(Z_z \setminus Z_\infty) \cdot U_\infty \to F_\infty$, defined by (2) is a homeomorphism. It can be checked without difficulty that F consists of the points of the form

$$x = x_0 z u. \tag{3}$$

† For the definition of Schwartz-Bruhat functions on G_A see **§ 3.5**.

Here $z \in Z_Q \setminus Z_A$, u ranges over the set U of elements of the form:

$$u = (u_\infty, u_2, \ldots, u_p, \ldots),$$

$$u_\infty \in U_\infty, \qquad u_p = \begin{pmatrix} a_p & b_p \\ (a_p^2 - \varepsilon_p b_p^2)^{-1}\varepsilon_p b_p & (a_p^2 - \varepsilon_p b_p^2)^{-1} a_p \end{pmatrix},$$

and $u_p \in U_p$, where U_p is the subgroup of integral matrices of G_p (see **§ 4.4**). We have to verify that the map $(Z_Q \setminus Z_A) \cdot U' \to F$ defined by (3) is one to one.

We assume that $x_0 zu = x_0 z'u'$. Then there exists a $\gamma_0 \in \Gamma$ such that $\gamma_0 zu = z'u'$. We represent z and z' in the form $z = \gamma z_1$, $z' = \gamma' z_1'$, where γ, $\gamma' \in Z_Q$, z_1, z_1' are elements of Z_A such that $(z_1)_p, (z_1')_p \in U_p \cap Z_p$, $p = 2, 3, \ldots$.

Then we have $\gamma'^{-1}\gamma_0\gamma z_1 u = z_1' u'$ and therefore,

$$\gamma'^{-1}\gamma_0\gamma(z_1 u)_p = (z_1' u')_p, \qquad p = 2, 3, \ldots.$$

Since $(z_1 u)_p, (z_1' u')_p \in U_p$, it follows that $\gamma'^{-1}\gamma_0\gamma \in G_Z$. But then the equation $\gamma'^{-1}\gamma_0\gamma(z_1 u)_\infty = (z_1' u')_\infty$ shows that $u_\infty = u_\infty'$; therefore, $\gamma'^{-1}\gamma_0\gamma \in Z_Q$. Since γ, $\gamma' \in Z_Q$, this proves that $\gamma_0 \in Z_Q$. Consequently, it follows from the equation $\gamma_0 zu = z'u'$ that z and z' belong to one and the same class in $Z_Q \setminus Z_A$ and that $u = u'$.

So we have shown that F is a cylindrical set.

We denote by H_E the collection of all functions in $L_2(X)$ that are concentrated on E, and by H_F^0 the collection of all functions in $L_2(X)$ that are concentrated on F and for which the integrals over the horospheres into which F is fibered are zero.

Obviously, $H^0 \subset H_E \dotplus H_F^0$. Therefore, the trace† of the operator T_φ on H^0 does not exceed the sum of the traces on H_E and H_F^0. The operator T_φ is positive-definite and an integral operator (see Chap. 1) with a kernel $K(x_1, x_2)$ that is bounded in every compact subdomain of X. Therefore, its trace on H_E is finite. The proof that the trace of T_φ on H_F^0 is finite proceeds just as in Chapter 1, **§ 6.5**.

7. The Spaces Y, Ω and E. The main object of study in the subsequent investigation is the spectrum of the representation in the space $L_2(X)/H^0$. Later, we shall see that the problem of the spectrum of $L_2(X)/H^0$ can be reduced to a simpler problem: the study of the representation in $L_2(\Omega)$, where $\Omega = D_Q Z_A \setminus G_A$. First, we turn to the solution of this simpler problem.

We begin with a description of the space $\gamma = Z_A \setminus G_A$, which

† By the trace of the operator T_φ on a subspace of H (which is, in general, not invariant under the operator T_φ) we understand the trace of the operator $PT_\varphi P$, where P is the projection operator onto the subspace of H (see Chap. 1, **§ 6**).

by analogy with the case of the group of real matrices of order 2 we call for convenience the *underlying affine space of* G_A.

We show that *the points of* Y *can be given by infinite sequences*

$$y = (y_\infty, y_2, \ldots, y_p, \ldots) \tag{1}$$

where $y_p = (y_p^1, y_p^2)$ *is a vector of the two-dimensional affine space over the field of p-adic numbers,* $y_p \neq 0$, *and from a sufficiently large* p *onwards, the numbers* y_p^1 *and* y_p^2 *are integers of which at least one is of norm* 1.

To see this, we associate with every element

$$g = (g_\infty, g_2, \ldots, g_p, \ldots)$$

of G_A the sequence

$$y = (y_\infty, y_2, \ldots, y_p, \ldots)$$

where y_p is the upper row of the matrix g_p, $p = \infty, 2, 3, \ldots$. Obviously, $g \to y$ is a map of G_A onto the space y of all sequences of the form (1). Let us find the stability group of Y. We fix in Y the point

$$y^0 = (y_\infty^0, y_2^0, \ldots, y_p^0, \ldots)$$

where $y_p^0 = (1, 0)$. Clearly, the stability group of y^0 is Z_A. Hence, $Y = Z_A \setminus G_A$. This proves the proposition.

From the description of the space Y it is clear that it is naturally embedded in the two-dimensional space A^2 over the group of adeles A. Moreover, it forms an everywhere dense subset in A^2, just as the group of ideles A^* forms an everywhere dense subset in A.

We show that Y *is a subspace of full measure in* A^2, *that is, for every measurable set* $F \subset A^2$ *the sets* F *and* $F \cap Y$ *have the same measure.*

Proof. We consider the subset $F^{(p_0)}$ of vectors of the form

$$y = (y_\infty, y_2, \ldots, y_p, \ldots),$$

where y_p for $p \leq p_0$ ranges over a measurable set in Q_p^2 of measure μ_p and $|y_p| \leq 1$ for $p > p_0$. Clearly, the union of these sets is the whole space A^2, hence it is sufficient to prove the proposition for these sets $F^{(p_0)}$.

Since the measure of the set of points $y_p \in Q_p^2$ for which $|y_p| \leq 1$ is 1, the measure $\mu(F^{(p_0)})$ of $F^{(p_0)}$ is $\mu_\infty \mu_2 \cdots \mu_{p_0}$. We now compute the measure $\mu(F^{(p_0)} \cap Y)$ of $F^{(p_0)} \cap Y$. We denote by $F_p^{(p_0)}$, $p > p_0$, the subset of vectors in $F^{(p_0)}$, with $|y_{p'}| = 1$ for $p' \geq p$. Since the measure of the set of points $y_{p'} \in Q_{p'}^2$ for which $|y_{p'}| = 1$ is $1 - \dfrac{1}{p'^2}$, we have

$$\mu(F_p^{(p_0)}) = \mu_\infty \mu_2 \cdots \mu_{p_0} \prod_{p' \geq p} \left(1 - \frac{1}{p'^2}\right)$$

$$= \mu(F^{(p_0)}) \prod_{p' \geq p} \left(1 - \frac{1}{p'^2}\right).$$

Obviously, the set $F^{(p_0)} \cap Y$ is the union of the sets $F_p^{(p_0)}$, and therefore

$$\mu(F^{(p_0)} \cap Y) = \lim_{p\to\infty} \mu(F_p^{(p_0)}) = \mu(F^{(p_0)}) \lim_{p\to\infty} \prod_{p'\geq p} \left(1 - \frac{1}{p'^2}\right).$$

From the fact that the product $\prod_p \left(1 - \frac{1}{p^2}\right)$ converges $\left(\text{it is equal to } \frac{1}{\zeta(2)} = \frac{6}{\pi^2}\right)$, it follows that $\lim_{p\to\infty} \prod_{p'\geq p} \left(1 - \frac{1}{p'^2}\right) = 1$. Thus, $\mu(F^{(p_0)} \cap Y) = \mu(F^{(p_0)})$, as required.

The group G_A acts in Y in the following way: the element $g = (g_\infty, g_2, \ldots, g_p, \ldots)$ of G_A carries the point

$$y = (y_\infty, y_2, \ldots, y_p, \ldots)$$

of Y into

$$yg = (y_\infty g_\infty, y_2 g_2, \ldots, y_p g_p, \ldots), \tag{2}$$

where $y_p g_p$ denotes the result of applying the matrix g_p to the row vector y_p.

Note that (2) defines the action of G_A on the whole space A^2. Relative to G_A this space splits into transitive parts, of which one is Y and the others are of the form aY, $a \in A$.

Now we establish a connection between Y and the space of horospheres Ω.

We define in Y an action of the idele group A^*. With every idele $\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots)$ we associate the transformation $y \to \lambda y$ in Y, which carries every point $y = (y_\infty, y_2, \ldots, y_p, \ldots)$ of Y into

$$\lambda y = (\lambda_\infty y_\infty, \lambda_2 y_2, \ldots, \lambda_p y_p, \ldots) \tag{3}$$

From the description of the space of horospheres Ω derived in **§ 4.3** it follows immediately that

$$\Omega = Q^* \setminus Y,$$

where Q^* is the subgroup of principal ideles. Thus, as a factor space of Y by a discrete subgroup, Ω is locally isomorphic to Y.

Let us show how to define the invariant integration on Y and Ω. Let dy_p be the invariant measure on the affine plane $y_p = (y_p^1, y_p^2)$, $p = \infty, 2, 3, \ldots$. For $p = 2, 3, \ldots$ we normalize it in the following way:

$$\int_{|y_p|\leq 1} dy_p = 1, \quad |y_p| = \max(|y_p'|_p, |y_p^2|_p). \tag{4}$$

It is easy to verify that the invariant integration on Y is expressed

by the following formula:

$$\int f(y)\,dy = \lim_{p\to\infty}\int f(y_\infty, y_2, \ldots, y_p, \ldots)\,dy_\infty\,dy_2\cdots dy_p. \quad (5)$$

Thus, the invariant measure dy on Y is expressed by the following symbolic formula:

$$dy = dy_\infty\,dy_2\cdots dy_p\cdots. \quad (6)$$

The invariant integration on Ω is defined by the same formula, because Ω is locally isomorphic to Y.

We give in Y a fundamental domain relative to the subgroup Q_+^* of positive rational numbers. We denote by E the subset of points

$$y = (y_\infty, y_2, \ldots, y_p, \ldots)$$

for which $|y_p| = 1$ for every prime $p = 2, 3, 5, \ldots$. Obviously, E is an open and closed subset of Y. Next we have

$$Y = \bigcup_{\lambda\in Q_+^*} \lambda E,$$

and the sets λE are pairwise disjoint. Consequently, E is a fundamental domain in Y relative to the subgroup Q_+^*. Thus,

$$E \cong Q_+^* \setminus Y.$$

If we identify in E the points y and $-y$, we obviously obtain a space homeomorphic to Ω. In what follows we shall frequently use this realization of the space Ω. In particular it is convenient to regard the function on Ω as functions $f(y)$ given on E and satisfying the condition $f(-y) = f(y)$.

8. The Operation of Multiplication in the Spaces A^2, Y, and E. We endow the space A^2 with the structure of a topological ring. We recall that A^2 is the set of all sequences

$$y = (y_\infty, y_2, \ldots, y_p, \ldots),$$

where $y_\infty \in R^2$, $y_p \in Q_p^2$, $p = 2, 3, \ldots$, and $|y_p| \leq 1$ for all p except a finite number.

Let $\varepsilon_p \in Q_p$ be an element of norm 1 that is not a square in Q_p, $p = 2, 3, \ldots$. We give isomorphisms of the additive groups:

$$R^2 \to C, \qquad Q_p^2 \to Q_p(\sqrt{\varepsilon_p})$$

in the following way:

$$y_\infty \equiv (y_\infty^{(1)}, y_\infty^{(2)}) \leftrightarrow y_\infty^{(1)} + \sqrt{-1}\,y_\infty^{(2)}, \quad (1)$$

$$y_p \equiv (y_p^{(1)}, y_p^{(2)}) \leftrightarrow y_p^{(1)} + \sqrt{\varepsilon_p}\,y_p^{(2)}. \quad (2)$$

Since C and $Q_p(\sqrt{\varepsilon_p})$ have the structure of a field, under the maps (1) and (2) this structure carries over to R^2 and Q_p^2.

We introduce the operation of multiplication in A^2 by defining the product of two elements in A^2 componentwise. Obviously, this operation is continuous in the topology of A^2. Thus, A^2 assumes the structure of a commutative topological ring.

We note that the isomorphisms of additive groups $R^2 \to C$ and $Q_p^2 \to Q_p(\sqrt{\varepsilon_p})$ can be given by various methods. Therefore, a multiplication in A^2 can also be introduced in other ways.

When A^2 is endowed with the structure of ring, the set of all invertible elements in A^2 coincides precisely with the subset Y. Thus, the space Y has the structure of a topological group.

In **§ 4.7** we introduced the subset $E \subset Y$ of elements

$$y = (y_\infty, y_2, \ldots, y_p, \ldots),$$

where $|y_2| = \cdots = |y_p| = \cdots = 1$. Obviously, E is a subgroup of Y in the sense of our operation of multiplication.

We indicate a convenient way of giving the elements $y \in E$. We denote by a_τ an element of E of the following form:

$$a_\tau = (y_\infty, y_2, \ldots, y_p, \ldots),$$

where

$$y_\infty = (\tau, 0), \quad \tau > 0,$$

$$y_p = (1, 0), \quad p = 2, 3, \ldots .$$

Next let $V \subset E$ be the subset of elements of the form

$$v = (v_\infty, v_2, \ldots, v_p, \ldots),$$

where

$$|v_\infty| = |v_2| = \cdots = |v_p| = \cdots = 1.$$

Observe that V is a compact subgroup.

It is easy to verify that every element $y \in Y$ can be represented in one and only one way as a product

$$y = a_\tau \circ v, \qquad v \in V.$$

The number $\tau > 0$ and the element $v \in V$ are called the polar coordinates of the point $y \in E$.

In polar coordinates the measure dy on the subset $E \subset Y$ is expressed by the following formula

$$dy = \tau \, d\tau \, dv,$$

where dv is the invariant measure on V.

The verification of this simple proposition is left to the reader.

We mention that apart from the operation of multiplication an operation of involution $y \to \bar{y}$ can be introduced in A^2. For if

$$y = (y_\infty, y_2, \ldots, y_p, \ldots), \quad y_p = (y_p^{(1)}, y_p^{(2)}), \quad p = \infty, 2, 3, \ldots,$$

we set

$$\bar{y} = (\bar{y}_\infty, \bar{y}_2, \ldots, \bar{y}_p, \ldots),$$

where

$$\bar{y}_p = (y_p^{(1)}, -y_p^{(2)}).$$

9. Decomposition of the Representations Generated by Y and Ω into Irreducible Representations. In **§ 4.5** we proved that the homogeneous spaces Y and Ω have an invariant measure. Consequently, in the spaces $L_2(Y)$ and $L_2(\Omega)$ the translation operators $T(g)f(x) = f(xg)$ define a unitary representation.

Our task is to decompose these representations into irreducible ones.

First we consider the space $L_2(Y)$. We take the set Π of all characters π of the idele group A^* (this set was described in **§ 1.9**). We set

$$\varphi_\pi(y) = \int_{A^*} f(\lambda y)\pi^{-1}(\lambda)\, |\lambda|\, d^*\lambda, \tag{1}$$

where $d^*\lambda$ is the invariant measure on the group A^* of all ideles† (see **§ 1.7**) and $f(y) \in L_2(Y)$.

It is not hard to see that the function φ_π satisfies the following condition of homogeneity:

$$\varphi_\pi(\lambda y) = \pi(\lambda)\, |\lambda|^{-1}\, \varphi_\pi(y), \quad \lambda \in A^*. \tag{2}$$

We can endow the set of functions satisfying (2) with the structure of a Hilbert space, by the same method as that used for the group of matrices of order 2 over a field. With this aim, we choose in Y a subset T having the following property: every point $y \in Y$ is uniquely representable in the form

$$y = \lambda t, \quad \lambda \in A^*, \quad t \in T. \tag{3}$$

In T we define a measure dt so that

$$dy = |\lambda|^2\, d^*\lambda\, dt. \tag{4}$$

In the set of functions on Y satisfying (2) we define a scalar product

$$(\varphi_\pi, \psi_\pi) = \int_T \varphi_\pi(t)\overline{\psi_\pi(t)}\, dt. \tag{5}$$

We denote the Hilbert space so obtained by H_π. An immediate

† We recall that $|\lambda| = |\lambda_\infty|_\infty\, |\lambda_2|_2 \cdots |\lambda_p|_p \cdots$

verification shows that the scalar product (5) is invariant under the operators $T_\pi(g)$:

$$T_\pi(g)\,\varphi_\pi(y) = \varphi_\pi(yg). \tag{6}$$

Thus, $T_\pi(g)$ is a unitary representation in H_π.

From results of **§ 3** it follows that *this representation is irreducible.* For let

$$\pi = (\pi_\infty, \pi_2, \ldots, \pi_p, \ldots).$$

Then our representation is, in the sense of **§ 3**, a tensor product of unitary representations $T_{\pi_p}(g_p)$ of the groups G_p, $p = \infty, 2, 3, \ldots,$ defined in the following way. The representation $T_{\pi_p}(g_p)$ is given in the space of functions $\varphi_{\pi_p}(y_p)$, $y_p = (y_p^1, y_p^2)$, satisfying the condition of homogeneity:

$$\varphi_{\pi_p}(\lambda_p y_p) = \pi_p(\lambda_p)\,|\lambda_p|_p^{-1}\,\varphi_{\pi_p}(y_p).$$

The representation operator $T_{\pi_p}(g_p)$ of G_p is given by the formula

$$T_{\pi_p}(g_p)\,\varphi_{\pi_p}(y_p) = \varphi_{\pi_p}(y_p g_p).$$

We know (see Chap. 2) that these representations of G_p are irreducible. Consequently, the representation $T_\pi(g)$ of G_A, as a tensor product of irreducible representations of the groups G_p, is itself irreducible.†

We call the representation $T_\pi(g)$ of G_A a *representation of the principal series* corresponding to the character π.

Note that by virtue of the condition of homogeneity (2) the representation space H_π can be realized as the space of functions $\varphi_\pi(t)$, $t \in T$, satisfying the condition

$$\|\varphi_\pi\|^2 = \int_T |\varphi_\pi(t)|^2\,dt < \infty.$$

The representation operator $T_\pi(g)$ is then given by the following formula:

$$T_\pi(g)\,\varphi_\pi(t) = \pi(\lambda')\,|\lambda'|^{-1}\,\varphi_\pi(t'),$$

where $\lambda' \in A^*$ and $t' \in T$ are defined by the relation $tg = \lambda' t'$.

Here is the final result.

The decomposition of the representation of G_A in $L_2(Y)$:

$$T(g)f(y) = f(yg) \tag{7}$$

into irreducible representations is realized by the following formulae:

$$f(y) = \int_\Pi \varphi_\pi(y)\,d\pi, \tag{8}$$

† More accurately, the representation T_{π_p} is irreducible only if $\pi_p \neq \mathrm{sign}_\tau$. However, the set of those $\pi \in \Pi$ for which $\pi_p = \mathrm{sign}_\tau$ for at least one p has measure zero.

where

$$\varphi_\pi(y) = \int_{A^*} f(\lambda y)\pi^{-1}(\lambda)\, |\lambda|\, d^*\lambda. \tag{9}$$

Furthermore, the Plancherel formula holds

$$\int_Y |f(y)|^2\, dy = \int_\Pi \|\varphi_\pi\|^2\, d\pi, \tag{10}$$

where $d\pi$ is the measure on the set Π of all characters of the idele group A^*, normalized so that

$$\int_\Pi \int_{A^*} u(\lambda)\pi(\lambda)\, d^*\lambda\, d\pi = u(1).$$

When the function $f(y)$ is transformed according to (7), its component $\varphi_\pi(t)$ is transformed according to the formula

$$T_\pi(g)\varphi_\pi(t) = \pi(\lambda')\, |\lambda'|^{-1}\, \varphi_\pi(t'), \tag{11}$$

where $\lambda' \in A^*$ and $t' \in T$ are defined by the relation $tg = \lambda' t'$. Thus, the irreducible unitary representation of the principal series corresponding to the character π acts in the space of functions $\varphi_\pi(t)$.

Proof. The decomposition (8) follows immediately (by the formula for the inverse Fourier transform) from the definition (9) of the function $\varphi_\pi(y)$. By the Plancherel formula for the Fourier transform it follows that

$$\int_{A^*} |f(\lambda t)|^2\, |\lambda|^2\, d^*\lambda = \int_\Pi |\varphi_\pi(t)|^2\, d\pi.$$

When both sides of this equation are integrated with respect to t, we obtain immediately the Plancherel formula (10). It remains to show that the functions $\varphi_\pi(t)$ are transformed according to (11). For this purpose we note that if $f(y)$ is transformed according to the formula $T(g)f(y) = f(yg)$, then $\varphi_\pi(y)$ is transformed according to the same formula

$$T(g)\varphi_\pi(y) = \varphi_\pi(yg).$$

On the other hand, $\varphi_\pi(y)$ satisfies the following condition of homogeneity:

$$\varphi_\pi(\lambda y) = \pi(\lambda)\, |\lambda|^{-1}\, \varphi_\pi(y).$$

Therefore, when we go over from the functions $\varphi_\pi(y)$ to their values $\varphi_\pi(t)$ on T, then the formula for $T(g)$ assumes the required form (11). The proof of the theorem is now complete.

Next we consider the problem of decomposing the representations in $L_2(\Omega)$ into irreducible representations.

First we note that functions on Ω can be regarded as functions on Y that are invariant under the transformations

$$f(y) \to f(\lambda y), \qquad \lambda \in Q^*,$$

where Q^* denotes the set of all principal ideles.

We denote by Π^* the set of all characters of the group A^*/Q^*. The following theorem holds.

The decomposition of the representation of G_A in $L_2(\Omega)$ into irreducible representations is realized by the following formulae:

$$f(y) = \int\limits_{\Pi^*} \varphi_\pi(y)\, d\pi, \tag{12}$$

where

$$\varphi_\pi(y) = \int\limits_{A^*/Q^*} f(\lambda y)\pi^{-1}(\lambda)\, |\lambda|\, d^*\lambda. \tag{13}$$

The Plancherel formula holds:

$$\int\limits_{Q^*\backslash Y} |f(y)|^2\, dy = \int\limits_{\Pi^*} \|\varphi_\pi\|^2\, d\pi, \tag{14}$$

where $d\pi$ is the measure on the set Π^* of all characters of A^*/Q^*, normalized so that

$$\int\limits_{\Pi^*} \int\limits_{A^*/Q^*} u(\lambda)\pi(\lambda)\, d\lambda^*\, d\pi = u(1).$$

The proof is similar to that of the preceding theorem.

Now let us find out with what multiplicity each representation of the fundamental series occurs in $L_2(Y)$ and $L_2(\Omega)$. In other words, we wish to determine which of the representations $T_\pi(g)$ occurring in the decomposition of $L_2(Y)$ and $L_2(\Omega)$ are equivalent.

Let $T_\pi(g)$ and $T_{\pi'}(g)$ be two irreducible representations of G_A corresponding, respectively, to the characters

$$\pi = (\pi_\infty, \pi_2, \ldots, \pi_p, \ldots) \quad \text{and} \quad \pi' = (\pi'_\infty, \pi'_2, \ldots, \pi'_p, \ldots). \tag{15}$$

According to **§ 3**, these representations are equivalent if and only if the corresponding representations of the groups G_p, $p = \infty, 2, 3, \ldots,$

$$T_{\pi_p}(g_p) \quad \text{and} \quad T_{\pi'_p}(g_p) \tag{16}$$

are equivalent.

But the representations (16) of G_p are equivalent if and only if $\pi'_p = \pi_p^{\varepsilon_p}$, where $\varepsilon_p = \pm 1$.

Thus, two representations $T_\pi(g)$ and $T_{\pi'}(g)$ of G_A are equivalent if and only if $\pi'_p = \pi_p^{\varepsilon_p}$, $\varepsilon_p = \pm 1$, $p = \infty, 2, 3, \ldots$.

Hence, we conclude at once that *every representation $T_\pi(g)$ occurs in $L_2(Y)$ with infinite multiplicity.*

For the space $L_2(\Omega)$ the picture is different. This space, as we know already, decomposes into representations $T_\pi(g)$, where π ranges over the set of characters that are equal to 1 on the subgroup of principal ideles.

We show that *two such characters π and π' give equivalent representations if and only if $\pi' = \pi^\varepsilon$, $\varepsilon = \pm 1$. Hence, every representation $T_\pi(g)$ occurs in $L_2(\Omega)$ with multiplicity* 2.

Suppose then that two characters π and π' that are equal to 1 on the group of principal ideles give equivalent representations.

As we know, the character π has the following form:

$$\pi(\lambda) = \pi_\infty(\lambda_\infty)\pi_2(\lambda_2)\ldots\pi_p(\lambda_p)\ldots,$$

where

$$\pi_\infty(\lambda_\infty) = |\lambda_\infty|_\infty^s \operatorname{sign}^\nu \lambda_\infty,$$

$$\pi_p(\lambda_p) = |\lambda_p|_p^{s_p}\,\theta_p(\lambda_p), \quad \theta_p(p) = 1, \quad p = 2, 3, 5, \ldots$$

Here s and s_p are imaginary numbers, $\nu = 0, 1$. Also, only finitely many characters θ_p are different from 1. Similarly,

$$\pi'(\lambda) = \pi'_\infty(\lambda_\infty)\pi'_2(\lambda_2)\ldots\pi'_p(\lambda_p)\ldots,$$

where

$$\pi'_\infty(\lambda_\infty) = |\lambda_\infty|_\infty^{s'} \operatorname{sign}^{\nu'} \lambda_\infty,$$

$$\pi'_p(\lambda_p) = |\lambda_p|^{s'_p}\,\theta'_p(\lambda_p), \quad \theta'_p(p) = 1.$$

The condition that equivalent representations correspond to π and π' yields

$$s' = \pm s, \quad \nu' = \nu,$$

$$s'_p = \varepsilon_p s_p, \quad \theta'_p = \theta_p^{\varepsilon_p},$$

where

$$\varepsilon_p = \pm 1, \quad p = 2, 3, \ldots.$$

Without loss of the generality of the arguments we may assume that $s' = s$. We show that then $\pi' = \pi$.

Since the character π is equal to 1 on the subgroup of principal ideles, $\pi(p) = 1$ for every prime number p, that is,

$$p^{s-s_p}\theta_2(p)\cdots\theta_p(p)\cdots = 1. \tag{17}$$

Observe that every character θ_p is of finite order and that among them only finitely many are different from 1. Hence, there exists an integer n such that $\theta_p^n = 1$ for every p. But then it follows from (17) that

$$p^{n(s-s_p)} = 1,$$

hence,

$$s - s_p = \frac{2\pi i}{n} k \ln p, \tag{18}$$

where k is an integer.

Similarly, we find

$$s - \varepsilon_p s_p = \frac{2\pi i}{n} k' \ln p \tag{19}$$

If $\varepsilon_p = -1$, then we obtain from (18) and (19)

$$s = \frac{\pi i}{n} k'' \ln p, \tag{20}$$

where k'' is an integer.

From (20) it follows that $\varepsilon_p = -1$ for at most one prime p. For if $\varepsilon_p = -1$ and $\varepsilon_q = -1$, $q \neq p$, then $\dfrac{\ln p}{\ln q} = r$, where r is a rational number; but this is impossible.

Thus, either $\varepsilon_p = 1$ for all p, or $\varepsilon_p = 1$ for all p except $p = p_0$. In the first case we have $\pi' = \pi$; in the second case we have $\pi'(\lambda) = \pi(\lambda)\pi_{p_0}^{-2}(\lambda_{p_0})$. But then, since $\pi(\lambda) = \pi'(\lambda) = 1$ on the subgroup of principal ideles, $\pi_{p_0}^2(r) = 1$ for every rational number $r \neq 0$. Since the rational numbers form an everywhere dense set in the multiplicative group of p-adic numbers, it follows that $\pi_p^2 = 1$. Therefore, in this case we also have $\pi(\lambda) = \pi'(\lambda)$, and the proposition is proved.

10. The Operator B (Definition). We introduce the operator

$$B: S(\Omega) \to C(\Omega)$$

in the space $S(\Omega)$ of all continuous rapidly decreasing functions on Ω, which maps $S(\Omega)$ into the space $C(\Omega)$ of continuous functions on Ω. This operator B will play a fundamental role in all our subsequent investigations.

We define B by the following formula:

$$(B\psi)(g) = \int_{Z_A} \psi(y_0 z g)\, dz, \quad g \in G_A, \tag{1}$$

where $y_0 = (0, 1)$. Obviously, the function $(B\psi)(g)$ is constant on the cosets $D_Q Z_A \setminus G_A$, and hence, may be regarded as a function on Ω. Thus, B carries a function on Ω again into a function on Ω. We call B the operator of the *horospherical automorphism* on Ω.

Observe that B commutes with the operators of group translation on Ω:

$$\psi(y) \to \psi(yg), \quad g \in G_A.$$

Later in **§ 4.11** we show that if ψ is a continuous fast decreasing

function on Ω, then the integral (1) necessarily converges for every $g \in G_A$ and so $B\psi$ is well-defined.

Next we obtain another equivalent expression for B.

First, we write $(B\psi)(y)$, $y \in Y$, instead of $(B\psi)(g)$. Since $(B\psi)(\lambda y) = (B\psi)(y)$ for every $\lambda \in Q^*$, we may always assume that $y \in E$. (We recall that $E \subset Y$ is the subset of points

$$y = (y_\infty, y_2, \ldots, y_p, \ldots)$$

for which $|y_2| = \cdots = |y_p| = \cdots = 1$.)

We write the expression for B in coordinate form

$$(B\psi)(y^{(1)}, y^{(2)}) = \int_A \psi(y^{(1)}t + y'^{(1)}, y^{(2)}t + y'^{(2)})\, dt, \tag{2}$$

where $y' = (y'^{(1)}y'^{(2)})$ is an arbitrary element of Y connected with y by the relation $y^{(1)}y'^{(2)} - y^{(2)}y'^{(1)} = 1$. The integral (2) does not depend on the choice of this element. Formula (2) can also be put in the following form, which is convenient for our purposes:

$$(B\psi)(y) = \int_{\substack{\langle y,z\rangle = 1,\\ z\in Y}} \psi(z)\omega_y(z), \tag{2'}$$

where $\langle y, z\rangle \equiv y^{(1)}z^{(2)} - y^{(2)}z^{(1)}$, and $\omega_y(z)$ is defined by the relation

$$dz^{(1)} \wedge dz^{(2)} - d\langle y, z\rangle \wedge \omega_y(z). \tag{3}$$

From this relation it follows that

$$\omega_y(z) = \frac{dz^{(1)}}{|y^{(1)}|} = \frac{dz^{(2)}}{|y^{(2)}|}. \tag{4}$$

An expression for $B\psi$ in which z ranges only over the points of E is specifically established in the following formula for B:

$$(B\psi)(y) = \sum_{n=1}^{\infty} \int_{\substack{\langle y,z\rangle = n,\\ z\in E}} \psi(z)\omega_y(z), \quad \text{where } y \in E. \tag{5}$$

For the proof we observe that Y is the union of the pairwise disjoint subsets $\lambda^{-1}E$, where λ ranges over the positive rational numbers. Consequently, the integral (2′) can be written as a sum of integrals over $\lambda^{-1}E$. We find:

$$\begin{aligned}(B\psi)(y) &= \sum_{\lambda} \int_{\substack{\langle y,z\rangle = 1,\\ z\in\lambda^{-1}E}} \psi(z)\omega_y(z)\\ &= \sum_{\lambda} \int_{\substack{\langle y,z\rangle = \lambda,\\ z\in E}} \psi(\lambda^{-1}z)\omega_y(\lambda^{-1}z) = \sum_{\lambda} \int_{\substack{\langle y,z\rangle = \lambda,\\ z\in E}} \psi(z)\omega_y(z). \end{aligned} \tag{6}$$

We have used the fact $\psi(\lambda^{-1}z) = \psi(z)$ and $\omega_y(\lambda^{-1}z) = \omega_y(z)$ for every $\lambda \in Q^*$.

It remains to observe that the summation in (6) is taken only over the (positive) integers λ. For since $y, z \in E$ we have $|y_p| = |z_p| = 1$ for every prime p, hence $|\langle y, z\rangle_p| \leq 1$ for every prime p. Consequently, λ is an integer.

Finally we write the expression for B in polar coordinates on E. We recall that every element $y \in E$ is uniquely representable in the form

$$y = a_\tau \circ v,$$

where a_τ, $v \in E$ are elements of the following form:

$$a_\tau = (\tilde{y}_\infty, \tilde{y}_2, \ldots, \tilde{y}_p, \ldots), \quad \tilde{y}_\infty = (\tau, 0), \quad \tau = |y|,$$

$$\tilde{y}_p = (1, 0), \quad p = 2, 3, \ldots;$$

$$v = (v_\infty, v_2, \ldots, v_p, \ldots), \quad |v_\infty| = |v_2| = \cdots = |v_p| = \cdots = 1.$$

The sign $\circ$ denotes the multiplication rule introduced in **§ 4.8.** By (5) we have

$$(B\psi)(a_\tau \circ v) = \sum_{n=1}^{\infty} \int_{\langle a_\tau \circ v, z\rangle = n} \psi(z)\omega_{a_\tau \circ v}(z) = \sum_{n=1}^{\infty} \int_{\langle a_\tau \circ v, z\circ \bar{v}^{-1}\rangle = n} \psi(z \circ \bar{v}^{-1})\omega_{a_\tau \circ v}(z \circ \bar{v}^{-1}).$$

Here $v \to \bar{v}$ is the involution introduced in **§ 4.8.**

From the definition of $\langle y, z\rangle$ and the measure $\omega_y(z)$ it follows immediately that

$$\langle a_\tau \circ v, z \circ \bar{v}^{-1}\rangle = \langle a_\tau, z\rangle = (\tau z_\infty^{(2)}, z_2^{(2)}, \ldots, z_p^{(2)}, \ldots),$$

$$\omega_{a_\tau \circ v}(z \circ \bar{v}^{-1}) = \omega_{a_\tau}(z) = \frac{dz^{(1)}}{\tau}.$$

Thus, the formula for B assumes the following form

$$(B\psi)(a_\tau \circ v) = \frac{1}{\tau} \sum_{n=1}^{\infty} \int_{\substack{\langle a_\tau, z\rangle = n, \\ z \in E}} \psi(z; v)\, dz^{(1)}$$

$$= \frac{1}{\tau} \sum_{n=1}^{\infty} \int \psi(z^{(1)}, n(a_z^{(1)})^{-1}; v)\, dz^{(1)}, \quad (7)$$

where

$$\psi(z^{(1)}, z^{(2)}; v) \equiv \psi(z; v) = \psi(z \circ \bar{v}^{-1}).$$

11. Properties of the Operator B. THEOREM 4. *Let $\psi(y)$ be a continuous function on Ω, fast decreasing as $|y| \to \infty$ and $|y| \to 0$.*† *Then*

† $|y| = |y_\infty|\,|y_2| \cdots |y_p| \cdots$. Observe that if $y \in E$, then $|y| = |y_\infty|$.

the operator B is defined for ψ, and $(B\psi)(y)$ is a continuous bounded function on Ω.

Proof. We estimate the expression for B under the integral sign in formula (7) of **§ 4.10.** From the fact that $\psi(z)$ decreases fast it follows that for every $N > 0$ we have the estimate

$$\psi(z; v) \equiv \psi(z^{(1)}, z^{(2)}; v) = O((1 + |z^{(1)}|)^{-N}(1 + |z^{(2)}|)^{-N}), \quad (1)$$

uniformly in v.

Consequently, the integral in (7) converges absolutely and uniformly, and

$$\int\limits_{\langle a_T, z\rangle = n} |\psi(z; v)|\, dz^{(1)} = O\left(\left(\frac{n}{|y|} + 1\right)^{-N}\right). \quad (2)$$

From the estimate (2) it follows that the series (7) converges absolutely and uniformly in y, when y ranges over any compact domain.

So we have shown that B is defined for ψ and that $(B\psi)(y)$ is a continuous function on Ω.

Now we show that $(B\psi)(y)$ is bounded. First of all, from (2) it follows that

$$(B\psi)(y) = O(|y|^{-N}) \quad \text{as } |y| \to 0$$

for every $N > 0$. In particular, this means that the function $(B\psi)(y)$ is bounded as $|y| \to 0$.

The fact that $(B\psi)(y)$ is bounded as $|y| \to \infty$ follows from the same estimate (2) and the fact that the sum $\frac{1}{\tau}\sum_{n=1}^{\infty}\left(\frac{n}{\tau} + 1\right)^{-N}$ is bounded as $\tau \to \infty$, if $N > 1$.

LEMMA 1. *For arbitrary rapidly decreasing functions ψ_1 and ψ_2 on Ω the equation*

$$(B\psi_1, \psi_2)_\Omega = (\psi_1, B\psi_2)_\Omega \quad (3)$$

holds, where

$$(\psi_1, \psi_2)_\Omega = \int\limits_\Omega \psi_1(y)\overline{\psi_2(y)}\, dy = \tfrac{1}{2}\int\limits_E \psi_1(y)\overline{\psi_2(y)}\, dy.$$

Proof. By the definition of the operator B we have

$$(B\psi_1, \psi_2)_\Omega = \tfrac{1}{2}\int\limits_E \left(\sum_{n=1}^{\infty} \int\limits_{\substack{\langle y,z\rangle = n,\\ z\in E}} \psi_1(z)\overline{\psi_2(y)}\omega_y(z)\right) dy. \quad (4)$$

Observe that in (4) we may interchange the summation over n and the integration over E. This results from the following estimate:

$$\int\limits_{\langle y,z\rangle = n} |\psi_1(z)\overline{\psi_2(y)}|\, \omega_y(z) = O(n^{-N}(1 + |y|)^{-N}), \quad (5)$$

which is valid for every $N > 0$.† So we have

$$(B\psi_1, \psi_2)_\Omega = \tfrac{1}{2} \sum_{n=1}^{\infty} \int_E \int_{\langle y,z\rangle = n} \psi_1(z)\overline{\psi_2(y)}\omega_y(z)\, dy.$$

Similarly:

$$(\psi_1, B\psi_2)_\Omega = \tfrac{1}{2} \sum_{n=1}^{\infty} \int_E \int_{\langle y,z\rangle = n} \psi_1(z)\overline{\psi_2(y)}\omega_z(y)\, dz.$$

Thus, to prove the lemma it is sufficient to verify that

$$\int_E \int_{\langle y,z\rangle = n} \psi_1(z)\overline{\psi_2(y)}\omega_y(z)\, dy = \int_E \int_{\langle y,z\rangle = n} \psi_1(z)\overline{\psi_2(y)}\omega_z(y)\, dz. \tag{6}$$

for every n. But the equation (6) is obvious, because each of the integrals in (6) is equal to

$$\int_{\substack{\langle y,z\rangle = n, \\ y,z\in E}} \psi_1(z)\overline{\psi_2(y)}\omega(y, z),$$

where $\omega(y, z)$ is defined by the relation

$$dy^{(1)} \wedge dy^{(2)} \wedge dz^{(1)} \wedge dz^{(2)} = d\langle y, z\rangle \wedge \omega(y, z).$$

Now we state the main theorem on the operator B.

THEOREM 5. *There exists a set ψ_c of rapidly decreasing functions on Ω that have the following properties:*

1. *ψ_c is everywhere dense in $L_2(\Omega)$.*
2. *$B\psi \in L_2(\Omega)$ for every function $\psi \in \Psi_c$.*
3. *The operator B can be extended from the subset Ψ_c to a unitary operator $\bar{B}$ in the space $L_2(\Omega)$. Here the operator $\bar{B}$ satisfies the relation:*

$$\bar{B}^2 = E \tag{7}$$

(*E is the unit operator*).

A proof of this theorem will be given in the next two subsections. Here we indicate only one important consequence of Theorem 5.

COROLLARY. *Is a fast decreasing function ϕ on Ω satisfies the condition $B\psi \in L_2(\Omega)$, then*

$$B\psi = \bar{B}\varphi.$$

It suffices to check that

$$(B\varphi, \psi)_\Omega = (\bar{B}\varphi, \psi)_\Omega \tag{8}$$

for every function $\psi \in \Psi_c$ (the parentheses denote the scalar product

† The estimate (5) is obtained in the following way. From (2) we see that $\int_{\langle y,z = n\rangle} |\psi_1(z)|\, \omega_y(z) = O\left(\frac{|y|^{N-1}}{n^N}\right)$. On the other hand, $\psi(y) = O((1 + |y|)^{2-N+1})$ from which (5) results immediately.

in $L_2(\Omega)$). But by Lemma 1 we have

$$(B\varphi, \psi)_\Omega = (\varphi, B\psi)_\Omega = (\varphi, \bar{B}\psi)_\Omega.$$

Furthermore, since $\bar{B}$ is a unitary operator and satisfies (7), we have

$$(\varphi, \bar{B}\psi)_\Omega = (\bar{B}\varphi, \psi)_\Omega,$$

and so (8) is proved.

12. Schwartz-Bruhat Functions in Ω. Here we give an effective construction of a family Ψ_c of functions on Ω that satisfy the requirements of Theorem 5.

As a preliminary we recall that Schwartz-Bruhat functions on the adele plane A^2 are finite linear combinations of functions of the form

$$\varphi(y) = \varphi_\infty(y_\infty)\varphi_2(y_2)\cdots\varphi_p(y_p)\cdots, \tag{1}$$

where $\varphi_\infty(y_\infty)$ is an infinitely differentiable rapidly decreasing function on the real plane R^2, $\varphi_p(y_p)$ a finite piecewise constant function on the p-adic plane Q_p^2, and all the functions $\varphi_p(y_p)$ except a finite number have the following form:

$$\varphi_p(y_p) = \begin{cases} 1, & \text{when } |y_p| \le 1, \\ 0, & \text{when } |y_p| > 1. \end{cases}$$

A Schwartz-Bruhat function is finite if and only if it is finite relative to y_∞.

We also recall that the underlying affine space Y is naturally embedded in A^2. It consists of all the elements

$$y = (y_\infty, y_2, \ldots, y_p, \ldots)$$

for which $y_\infty \neq 0$, $y_p \neq 0$, $p = 2, 3, \ldots$, and $|y_p| = 1$ for all p except a finite number.

Lemma 2. *Let $\varphi(y)$ be a Schwartz-Bruhat function on A^2. Then:*

1. *The series*

$$\psi(y) = \sum_{\lambda\in Q^*} \varphi(\lambda y), \quad y \in Y \tag{2}$$

converges absolutely and uniformly in every compact domain of Y. Thus, $\psi(y)$ is a continuous function of y.

2. *For every $N > 0$ the following estimate holds:*

$$\psi(y) = O(|y|^{-N}) \quad \text{as} \quad |y| \to \infty; \tag{3}$$

if $\varphi(y)$ is a finite function, then $\psi(y)$ is concentrated in the domain $|y| < C$.

Proof. We may assume that $y \in E$, where E is the domain defined in **§ 4.7**. If $\varphi(\lambda y) \neq 0$, $y \in E$, then by the definition of a Schwartz-Bruhat function, $|\lambda y_p| = 1$ for all p beginning with a certain p_0, and $|\lambda y_p| \le C_p$ for $p < p_0$, where c_p are certain constants.

Since it follows from $y \in E$ that $|y_p| = 1$ for all p, we have: $|\lambda|_p = 1$ for $p \geq p_0$, $|\lambda|_p \leq C_p$ for $p < p_0$. Hence, it follows that the summation in (2), in fact, is taken only over the numbers of the form $\lambda = \dfrac{n}{n_0}$, where $n = \pm 1, \pm 2, \ldots$, and the denominator n_0 is fixed:

$$\psi(y) = \sum_{n \neq 0} \varphi\left(\frac{n}{n_0} y\right). \tag{2'}$$

From the fact that φ is a rapidly decreasing function of y_∞, we have for $y \in E$:

$$\left| \varphi\left(\frac{n}{n_0} y\right) \right| < C_N \, |n|_\infty^{-N} \, |y|^{-N}.$$

From this estimate it follows that the series (2′) converges absolutely and uniformly on every compact set and that the estimate (3) holds for $\psi(y)$. Finally, if $\varphi(y)$ is a finite function, then there exists a C such that $\varphi(y) = 0$ when $|y_\infty| > C$. Therefore, if $|y| \equiv |y_\infty| > n_0 C$, all the terms in (2′) are zero. Hence, $\psi(y) = 0$.

The function $\psi(y)$ defined by (2) obviously satisfies the condition $\psi(\lambda y) = \psi(y)$ for every $\lambda \in Q^*$. Therefore, it may be regarded as a function on $\Omega = Q^* \setminus Y$.

The functions $\psi(y)$, $y \in Y$, that are representable in the form (2), where $\varphi(y)$ is a Schwartz-Bruhat function on A^2, are called Schwartz-Bruhat functions on Ω.

Lemma 3. *For a Schwartz-Bruhat function $\psi(y)$ on Ω,*

$$\psi(y) = \sum_{\lambda \in Q^*} \varphi(\lambda y)$$

the following estimate holds as $|y| \to 0$:

$$\psi(y) = -\varphi(0) + |y|^{-1} \int_A \varphi(ty)\, dt + O(|y|^{-N}), \tag{4}$$

where N is any positive integer.

Proof. We represent the elements $y \in Y$ in the form $y = tv$, where $t \in A^*$ and v belongs to the compact set $V \subset Y$ defined by the condition

$$|v_\infty| = |v_2| = \cdots = |v_p| = \cdots = 1.$$

For every fixed v the function $\varphi(t, v) = \varphi(tv)$ is a Schwartz-Bruhat function of $t \in A$, therefore the Poisson formula is applicable to it. We denote by $\tilde{\varphi}(t, v)$ the Fourier transform of $\varphi(t, v)$ with respect to t. Observing that

$$\tilde{\varphi}(0, v) = \int_A \varphi(tv)\, dt,$$

we find by the Poisson formula:

$$\psi(tv) = -\varphi(0) + |t|^{-1}\int_A \varphi(tv)\,dt + |t|^{-1}\sum_{\lambda\in Q^\bullet}\tilde{\varphi}(\lambda t^{-1}, v).$$

$\tilde{\varphi}(t, v)$ is a rapidly decreasing function of t, uniformly in v. Therefore, by the same arguments as in the proof of Lemma 2 we obtain the following estimate:

$$|t|^{-1}\sum_{\lambda\in Q^\bullet}\tilde{\varphi}(\lambda t^{-1}, v) = \mathrm{O}(|t|^{-N}) \quad \text{as } |t| \to 0.$$

Thus,

$$\psi(tv) = -\varphi(0) + |t|^{-1}\int_A \varphi(tv)\,dt + \mathrm{O}(|t|^{-N}),$$

uniformly in v. Since for every $y = tv \in E$ we have $|y| = |t|$, this equation is equivalent to (4).

DEFINITION. *Let Φ be the subset of Schwartz-Bruhat functions on A^2 that satisfy the following additional conditions:*

1. $\int_A \varphi(ty)\,dt = 0$ *for every* $y \in Y$;
2. $\varphi(0) = 0$;
3. $\int_Y \varphi(y)\,dy = 0$.

We introduce the set Ψ of functions of the form

$$\psi(y) = \sum_{\lambda\in Q^\bullet}\varphi(\lambda y), \quad \text{where } \varphi \in \Phi. \tag{5}$$

By Ψ_c we denote the subset of functions of the form (5), where $\varphi \in \Phi$ satisfies the additional condition of finiteness.

In this and the following subsection we prove that the family of functions Ψ_c satisfies all the requirements of Theorem 5.

First of all, we observe that the functions $\psi \in \Psi$ are continuous and decrease rapidly on Ω (that is, for every $N > 0$ we have

$$\psi(y) = \mathrm{O}(|y|^{-N}) \quad \text{as } |y| \to \infty$$

and

$$\psi(y) = \mathrm{O}(|y|^{N}) \quad \text{as } |y| \to 0).$$

This follows immediately from the estimates obtained in Lemmas 2 and 3 and from condition 1 and 2 on $\varphi(y)$.

LEMMA 4. *The set Ψ_c is everywhere dense in $L_2(\Omega)$.*

Proof. Suppose the contrary. Then there exists a function $f \in L_2(\Omega)$, $f \neq 0$, such that

$$\int_E f(y)\overline{\psi(y)}\,dy = 0 \tag{6}$$

for every function $\psi \in \Psi_c$. We now reduce (6) to the analogous condition for functions of a single real variable.

First we endow A^2 with the structure of a ring. For this purpose we realize A^2 as the set of elements of the form

$$y = (y_\infty, y_2, \ldots, y_p, \ldots),$$

where

$$y_\infty \in C, \quad y_p \in Q_p(\sqrt{\varepsilon_p}), \quad p = 2, 3, \ldots,$$

and $|y_p| \leq 1$ for all p except a finite number (see **§ 4.8**). Then every element $y \in A^2$ can be written as a product

$$y = a_\tau \circ u,$$

where

$$a_\tau = (\tau, 1, \ldots, 1, \ldots), \quad \tau = |y_\infty|; \quad u = (\tau^{-1} y_\infty, y_2, \ldots, y_p, \ldots).$$

In particular, the elements $y \in E$ have the form

$$y = a_\tau \circ v,$$

where

$$v = (v_\infty, v_2, \ldots, v_p, \ldots), \qquad |v_\infty| = |v_2| = \cdots = |v_p| = \cdots = 1. \tag{7}$$

We note that the elements v of the form (7) form a compact group (under multiplication), which we denote by V.

Since by assumption $f(y) \neq 0$, there exists a character $\pi(v)$ on V such that

$$F(\tau) = \int_V f(a_\tau \circ v) \pi^{-1}(v) \, dv \not\equiv 0 \tag{8}$$

The integration is taken with respect to the invariant measure dv on V.

We mention two properties of $\pi(v)$.

1. $\pi(v)$ can be expressed by the following formula:

$$\pi(v) = \pi_\infty(v_\infty) \prod_{p \leq p_0} \pi_p(v_p), \tag{9}$$

where $\pi_\infty(v_\infty) \pi_p$ and (v_p) are, respectively, characters on the subgroups of elements $v_\infty \in C$, $|v_\infty| = 1$, and $v_p \in Q_p(\sqrt{\varepsilon_p})$. The product is taken over all primes p not exceeding a certain p_0.

2. $$\pi(-1) = 1. \tag{10}$$

The first property follows immediately from the continuity of $\pi(v)$. The second from the fact that $f \in L_2(\Omega)$ satisfies on E the condition $f(-y) = f(y)$; hence, $f(a_z \circ v) = f(a_z \circ (-v))$.

With the character π we associate the family $\Phi^{(\pi)} \subset \Phi$ of finite Schwartz-Bruhat functions on A^2. For this purpose we take

on the half-line $0 \leq \tau < \infty$ the family Φ_∞ of functions $\varphi_\infty(\tau)$ satisfying the following conditions:

1. $\varphi_\infty(\tau)$ is a finite, infinitely differentiable function and is zero in the neighborhood of 0.

2. $\int_0^\infty \varphi_\infty(\tau)\, d\tau = 0;$

3. $\int_0^\infty \varphi_\infty(\tau)\tau\, d\tau = 0.$

Let $\Phi^{(\pi)}$ be the set of functions on A^2 defined in the following way:

$$\varphi(y) \equiv \varphi(a_z \circ u) = \varphi_\infty(\tau)\pi_\infty(u_\infty) \prod_{p \leq p_0} \pi_p(u_p), \quad \varphi_\infty \in \Phi_\infty,$$

when $|u_2| = \cdots = |u_{p_0}| = 1$ and $|u_p| \leq 1$ for $p > p_0$; $\varphi(a_z \circ u) = 0$ for all remaining u.

From the definition it follows immediately that the functions $\varphi \in \Phi^{(\pi)}$ are finite Schwartz-Bruhat functions on A^2 and that they belong to the family Φ.

For the functions $\varphi \in \Phi^{(\pi)}$ we construct $\varphi(y) \in \Psi_c$:

$$\varphi(y) = \sum_{\lambda \in Q^*} \varphi(\lambda y), \quad y \in E \tag{11}$$

We now find an explicit expression for $\psi(y)$ in terms of $\varphi_\infty(\tau)$.

We show that $\psi(y)$ *can be expressed by the following formula:*

$$\psi(a_z \circ v) = \psi_\infty(\tau)\pi(v), \tag{12}$$

where

$$\psi_\infty(\tau) = 2 \sum_{n=1}^{\infty} \theta(n)\, \varphi_\infty(n\tau), \quad \varphi_\infty \in \Phi_\infty; \tag{13}$$

$\theta(n)$ is a numerical character, that is, a function satisfying the following conditions:

1. $\theta(1) = 1.$
2. $\theta(n_1 n_2) = \theta(n_1)\theta(n_2)$ for any natural numbers n_1 and n_2.
3. There exists an integer q such that $\theta(n + q) = \theta(n)$.

This numerical character $\theta(n)$ is completely determined by $\pi(v)$.

As a preliminary we observe that the summation in (11) is taken only over integers n that are coprime to the numbers $2, 3, \ldots, p_0$.

For if $\varphi(\lambda y) \neq 0$, then by definition of $\varphi(y)$ we have $|\lambda y_p| = 1$ for $p \leq p_0$ and $|\lambda y_p| \leq 1$ for $p > p_0$. Since $y \in E$, we have $|y_p| = 1$

for all p; consequently, $|\lambda|_p = 1$ for $p \leq p_0$, $|\lambda|_p \leq 1$ for $p > p_0$. But this means that λ is an integer coprime to $2, 3, \ldots p_0$.

So we have:

$$\psi(y) = \sum_{n \neq 0}' \varphi(ny),$$

where the summation is taken only over the integers n that are coprime to $2, 3, \ldots, p_0$.

Next we represent the elements $y \in E$ in the form $y = a_z \circ v$. Then we have

$$ny = a_{|n|\tau} \circ v^{(n)},$$

where

$$v^{(n)} = ((\operatorname{sign} n)v_\infty, nv_2, \ldots, nv_p, \ldots).$$

Consequently,

$$\varphi(ny) = \varphi_\infty(|n|\tau)\theta(n)\pi(v),$$

where

$$\theta(n) = \pi_\infty(\operatorname{sign} n)\pi_2(n) \cdots \pi_{p_0}(n).$$

Note that $\theta(-1) = 1$, because $\pi(-1) = 1$.

We extend $\theta(n)$ to the set of all integers $n \neq 0$, by setting $\theta(n) = 0$ when n is divisible by at least one of the numbers $2, 3, \ldots, p_0$.

From the definition of $\theta(n)$ and the periodicity of the characters $\pi_p(v_p)$ it follows that $\theta(n)$ is a numerical character. As a result we obtain the required expression for $\psi(y)$:

$$\psi(a_\tau \circ v) = \psi_\infty(\tau)\pi(v),$$

where

$$\psi(\tau) = 2\sum_{n=1}^{\infty} \theta(n)\varphi_\infty(n\tau), \quad \varphi_\infty(\tau) \in \Phi_\infty.$$

Now we substitute a function $\psi \in \Psi_c$ of the form (12) in (6). We obtain:

$$\int_0^\infty F(\tau)\overline{\psi_\infty(\tau)}\tau \, d\tau = 0, \tag{14}$$

where

$$F(\tau) = \int_V f(a_\tau \circ v)\pi^{-1}(v) \, dv \not\equiv 0. \tag{15}$$

So we have reduced the original condition (6) on the function $f \in L_2(\Omega)$ to the condition (14) for a function of the real variable τ. Here we have made use of the following lemma, which will be proved in **§ 4** Appendix **1.1**.

Lemma. *Let Φ_∞ be the family of functions defined on the half-line $0 \leq \tau < \infty$ and satisfying conditions* 1–3 (*see p.* 315). *If the function $F(\tau)$*

is such that

$$\int_0^\infty |F(\tau)|^2 \tau \, d\tau < +\infty,$$

and $F(\tau)$ *satisfies* (14) *for every finite function* $\psi_\infty(\tau)$ *of the form* (13), *then* $F(\tau) \equiv 0$.

Since the function $F(\tau)$ defined by (15) satisfies the conditions of the lemma, $F(\tau) \equiv 0$. But this contradicts the assumption that $F(\tau) \not\equiv 0$, and Lemma 4 is proved.

13. The Fourier Transform in $L_2(\Omega)$. In the preceding subsection we introduced the family Ψ_c of rapidly decreasing functions on Ω showing that it has the first of the properties stated in Theorem 5: Ψ_c is everywhere dense in $L_2(\Omega)$. Now we prove that Ψ_c has the other two properties, namely: $B\psi \in L_2(\Omega)$ for every $\psi \in \Psi_c$; and, the operator B extends from the subset Ψ_c to a unitary operator $\bar{B}$ defined on $L_2(\Omega)$ and satisfies the relation $\bar{B}^2 = E$.

With this aim we now introduce an operator

$$F: \Psi \to \Psi.$$

It will be shown that F extends to a unitary operator, defined on the whole of $L_2(\Omega)$ and satisfying the relation $F^2 = E$ (Theorem 6). We call the unitary operator F the Fourier transform in $L_2(\Omega)$.

Furthermore, it will be shown that $B\psi = F\psi$ for every function $\psi \in \Psi_c$ (Theorem 7). The required property of the family Ψ_c follows immediately from this.

First, we define the Fourier transform in the space of functions on A^2 by the following formula

$$\tilde{\varphi}(y^{(1)}, y^{(2)}) = \int \varphi(z^{(1)}, z^{(2)}) \chi_0(z^{(1)}y^{(2)} - z^{(2)}y^{(1)}) \, dz^{(1)} \, dz^{(2)} \tag{1}$$

Here $\chi_0(t)$ is the additive character on A described in **§ 1.5.**

In **§ 2.5** the Fourier transform was defined by another formula:

$$\tilde{\varphi}(y^{(1)}, y^{(2)}) = \int \varphi(z^{(1)}, z^{(2)}) \chi_0(z^{(1)}y^{(1)} + z^{(2)}y^{(2)}) \, dz^{(1)} \, dz^{(2)}.$$

Clearly, the basic properties of the Fourier transform indicated there remain valid with the new definition.

The advantage of the new definition of the Fourier transform is that this transform commutes with the operators of group translation; that is,

$$\tilde{\varphi}(yg) = \widetilde{\varphi(yg)}, \quad g \in G_A \tag{2}$$

whereas the usual definition of the Fourier transform leads to the

more complicated relation $\tilde{\varphi}(yg) = \widetilde{\varphi(yg'^{-1})}$, where g' is the transpose of the matrix g. Furthermore, with the new definition of a Fourier transform we have

$$\tilde{\tilde{\varphi}}(y) = \varphi(y), \tag{3}$$

that is, the square of the operator of Fourier transformation is the unit operator; whereas for the usual Fourier transform

$$\tilde{\tilde{\varphi}}(y) = \varphi(-y).$$

LEMMA 5. *The Fourier transform carries the subset Φ of Schwartz-Bruhat functions defined in* **§ 4.12** *into itself.*

For we know that the Fourier transform of a Schwartz-Bruhat function is again a Schwartz-Bruhat function. Next, we show that if $\int \varphi(ty)\,dt = 0$, then $\int \tilde{\varphi}(ty)\,dt = 0$. It is sufficient to prove the proposition for an arbitrary fixed y, for example, $y = (0, 1)$. For $y = (0, 1)$ we have:

$$\int \tilde{\varphi}(ty)\,dt = \int \tilde{\varphi}(0, t)\,dt = \hat{\varphi}(0, 0),$$

where

$$\hat{\varphi}(y^{(1)}, y^{(2)}) = \int \tilde{\varphi}(y^{(2)}, t)\chi_0(-ty^{(1)})\,dt.$$

We express $\hat{\varphi}$ in terms of the function $\varphi(y^{(1)}, y^{(2)})$. Since

$$\int \hat{\varphi}(y^{(1)}, s)\chi_0(sy^{(2)})\,ds = \int \tilde{\varphi}(s, t)\chi_0(sy^{(2)} - ty^{(1)})\,ds\,dt = \varphi(y^{(1)}, y^{(2)}),$$

we have by the inversion formula for the Fourier transform

$$\hat{\varphi}(y^{(1)}, s) = \int \varphi(y^{(1)}, y^{(2)})\chi_0(-sy^{(2)})\,dy^{(2)}.$$

Hence,

$$\hat{\varphi}(0, 0) = \int \varphi(0, y^{(2)})\,dy^{(2)} = 0.$$

So we have shown that

$$\int \tilde{\varphi}(ty)\,dt = 0.$$

Finally, we note that condition 3 on a function $\varphi \in \Phi$ may be written in the form $\hat{\varphi}(0, 0) = 0$. Thus, in the transition from φ to its Fourier transform $\tilde{\varphi}$ conditions 2 and 3 are simply interchanged.

Now we define the Fourier transform in the space $L_2(\Omega)$.

First we define it on the subset of functions $\psi \in \Psi$, that is, functions of the form

We define the Fourier transform of a function $\psi(y)$ of the form (4) by the formula:

$$\tilde{\psi}(y) \equiv (F\psi)(y) = \sum_{\lambda \in Q^*} \tilde{\varphi}(\lambda y), \quad y \in Y, \tag{5}$$

where $\tilde{\varphi}(y)$ is the Fourier transform of $\varphi \in \Phi$. By what we said previously on the Fourier transform of functions $\varphi \in \Phi$, the function $F(\psi)(y)$ also belongs to Ψ.

THEOREM 6. *For any two functions* $\Psi_1(y), \Psi_2(y) \in \Psi$ *the equation*

$$(F\psi_1, F\psi_2)_\Omega = (\psi_1, \psi_2)\Omega \tag{6}$$

holds, where $(\ , \ )\Omega$ *is the scalar product in* $L_2(\Omega)$. *Thus, since* Ψ *is everywhere dense in* $L_2(\Omega)$, *the operator* F *extends in a unique way from* Ψ *to a unitary operator in* $L_2(\Omega)$.

We reduce the proof of Theorem 6 to two propositions.

PROPOSITION 1. *Let*

$$\psi_1(y) = \sum_{\lambda \in Q^*} \varphi_1(\lambda y), \quad \varphi_1 \in \Phi, \tag{7}$$

and let $\psi_2(y)$ *be any bounded function on* Ω. *Then*

$$\int_E \psi_1(y)\overline{\psi_2(y)}\, dy = \sum_{\lambda \in Q^*} \int_E \varphi_1(\lambda y)\overline{\psi_2(y)}\, dy, \tag{8}$$

where the series on the right converges absolutely.

Proof. As we remarked in the proof of Lemma 2, the summation in (7) is taken, in fact, over the numbers of the form $\lambda = \dfrac{n}{n_0}$ where n_0 is fixed and $n = \pm 1, \pm 2, \ldots$. We split the integral (8) into a sum of two integrals:

$$\int_E \psi_1(y)\overline{\psi_2(y)}\, dy = \int_{\substack{E,\\ |y| \geq 1}} \left(\sum_{n \neq 0} \varphi_1\left(\frac{n}{n_0} y\right) \overline{\psi_2(y)} \right) dy + \int_{\substack{E,\\ |y| < 1}} \left(\sum_{n \neq 0} \varphi_1\left(\frac{n}{n_0} y\right) \overline{\psi_2(y)} \right) dy.$$

We have to check that in each of these integrals summation and integration can be interchanged, and that the series so obtained converge absolutely.

For the first integral this follows immediately from the estimate

$$\varphi_1\left(\frac{n}{n_0} y\right)\overline{\psi_2(y)} = O(|n|_\infty^{-N} |y|^{-N})$$

for $N > 2$; for the second integral it follows from the same estimate for $1 < N < 2$. Here we have to bear in mind the expression for the

measure dy in "polar coordinates." For if we represent the elements $y \in E$ in the form of a product $y = a_\tau \circ v$, where

$$a_\tau = (\tilde{y}_\infty, \tilde{y}_2, \ldots, \tilde{y}_p, \ldots), \tilde{y}_\infty = (\tau, 0), \tau = |y|,$$
$$\tilde{y}_p = (1, 0), p = 2, 3, \ldots ; v = (v_\infty, v_2, \ldots, v_p, \ldots),$$

$|v_\infty| = |v_2| = \cdots = |v_p| = \cdots = 1$, then the measure dy is expressed by the following formula: $dy = \tau\, d\tau\, dv$.

PROPOSITION 2. *If* $\psi_1(y) = \sum_{\lambda \in Q^*} \varphi_1(\lambda y)$,

$$\psi_2(y) = \sum_{\lambda \in Q^*} \varphi_2(\lambda y)$$

where $\varphi_1, \varphi_2 \in \Phi$, *then*

$$\int_E \psi_1(y)\overline{\psi_2(y)}\, dy = \sum_{\lambda, \mu \in Q^*} \int_E \varphi_1(\lambda y)\overline{\varphi_2(\mu y)}\, dy, \tag{9}$$

and the series on the right converges absolutely.

Proof. By Proposition 1 we have

$$\int_E \psi_1(y)\overline{\psi_2(y)}\, dy = \sum_{\lambda \in Q^*} \int_E \varphi_1(\lambda y)\overline{\psi_2(y)}\, dy, \tag{10}$$

and the series on the right converges absolutely.

Again applying Proposition 1 we find:

$$\int_E \varphi_1(\lambda y)\overline{\psi_2(y)}\, dy = \sum_{\mu \in Q^*} \int_E \varphi_1(\lambda y)\overline{\varphi_2(\mu y)}\, dy, \tag{11}$$

and the integral on the right converges absolutely. Proposition 2 now follows from equations (10) and (11).

Proof of Theorem 6. We make use of the equation (9) from which it follows that

$$\int_E \psi_1(y)\overline{\psi_2(y)}\, dy = 2 \sum_{\mu \in Q^*} \int_Y \varphi_1(y)\overline{\varphi_2(\mu y)}\, dy. \tag{12}$$

For

$$\sum_{\lambda, \mu \in Q^*} \int_E \varphi_1(\lambda y)\overline{\varphi_2(\mu y)}\, dy = \sum_{\lambda, \mu \in Q^*} \int_{\lambda E} \varphi_1(y)\overline{\varphi_2(\mu \lambda^{-1} y)}\, dy$$
$$= \sum_{\lambda, \mu \in Q^*} \int_{\lambda E} \varphi_1(y)\overline{\varphi_2(\mu y)}\, dy = 2 \sum_{\mu \in Q^*} \int_Y \varphi_1(y)\overline{\varphi_2(\mu y)}\, dy.\dagger$$

Similarly we have

$$\int_E \tilde{\psi}_1(y)\overline{\tilde{\psi}_2(y)}\, dy = 2 \sum_{\mu \in Q^*} \int_Y \tilde{\varphi}_1(y)\overline{\varphi_2(\mu y)}\, dy. \tag{13}$$

† We use the fact that when λ ranges over Q^*, the set λE covers Y *twice*.

But since $\varphi(\mu y) = \tilde{\varphi}(\mu^{-1}y)$, by the Plancherel formula on Y we have:

$$\int_Y \varphi_1(y)\overline{\varphi_2(\mu y)}\,dy = \int_Y \tilde{\varphi}_1(y)\overline{\tilde{\varphi}_2(\mu^{-1}y)}\,dy.$$

Consequently, the right-hand sides of (12) and (13) are the same, and so equation (6) is proved.

From Theorem 6 it follows, in particular, that the Fourier transform of a function $\psi \in \Psi$ is uniquely determined by (5), that is, it does not depend on the method of representing $\psi(y)$ in the form (4).

COROLLARY. *The operator F on $L_2(\Omega)$ satisfies the relation*

$$F^2 = E, \tag{14}$$

where E is the unit operator.

For if $\psi \in \Psi$, then the equation $F^2\psi = \psi$ immediately follows from the definition of F on Ψ. Since Ψ is everywhere dense in $L_2(\Omega)$, (14) is valid on the whole of $L_2(\Omega)$.

THEOREM 7. $F\psi = B\psi$ *for every Schwartz-Bruhat function* $\psi \in \Psi_c$.

Proof. Since the group G_A acts transitively in Ω and the operators F and B commute with the operators of group translation, it is enough to prove the equation $(F\psi)(y) = (B\psi)(y)$ for an arbitrary individual point y, for example, for $y_0 = (0, 1)$.

By definition,

$$\psi(y) = \sum_{\lambda\in Q^*} \varphi(\lambda y), \tag{15}$$

$$(F\psi)(y) = \sum_{\lambda\in Q^*} \tilde{\varphi}(\lambda y), \tag{16}$$

where $\varphi(y)$ is a finite Schwartz-Bruhat function on A^2, $\varphi \in \Phi$, and $\tilde{\varphi}(y)$ its Fourier transform. Consequently,

$$(F\psi)(0, 1) = \sum_{\lambda\in Q^*} \tilde{\varphi}(0, \lambda).$$

Since $\tilde{\varphi}(0, t)$ is a Schwartz-Bruhat function on A, we may apply the Poisson formula to this expression. As a result we obtain

$$(F\psi)(0, 1) = \sum_{\lambda\in Q^*} \hat{\varphi}(\lambda, 0) + \hat{\varphi}(0, 0) - \tilde{\varphi}(0, 0), \tag{17}$$

where $\hat{\varphi}$ denotes the Fourier transform of $\tilde{\varphi}$ with respect to the second argument:

$$\hat{\varphi}(y^{(1)}, y^{(2)}) = \int \tilde{\varphi}(y^{(2)}, t)\chi_0(-ty^{(1)})\,dt.$$

But this function $\hat{\varphi}(y^{(1)}, y^{(2)})$ is expressed in the following way in

terms of the original function $\varphi(y^{(1)}, y^{(2)})$:

$$\hat{\varphi}(y^{(1)}, s) = \int \varphi(y^{(1)}, y^{(2)}) \chi_0(-sy^{(2)})\, dy^{(2)}.$$

Thus, equation (17) assumes the following form:

$$(F\psi)(0, 1) = \sum_{\lambda \in Q^*} \int_A \varphi(\lambda, t)\, dt + \int_A \varphi(0, t)\, dt - \tilde{\varphi}(0, 0). \quad (18)$$

Since $\varphi \in \Phi$, we have

$$\tilde{\varphi}(0, 0) = 0, \qquad \int_A \varphi(0, t)\, dt = 0.$$

After the change of variable $t \to \lambda t$ we may rewrite (18) in the following form:

$$(F\psi)(0, 1) = \sum_{\lambda \in Q^*} \int_A \varphi(\lambda, \lambda t)\, dt. \quad (19)$$

Observe that here the summation is taken, in fact, only over a finite set of values λ so that in (19) summation and integration may be interchanged. Indeed, from the definition of a Schwartz-Bruhat function it follows that if $\varphi(\lambda, \lambda t) \neq 0$, then $|\lambda|_p \leq 1$ for $p \geq p_0$, $|\lambda|_p < c_p$ for $p < p_0$. Here the prime number p_0 and the constants c_p depend only on the function φ. Hence, it follows that λ only assumes the values $\lambda = \dfrac{n}{n_0}$, where n_0 is fixed and $n = \pm 1, \pm 2, \ldots$. On the other hand, since φ is a finite function, it follows from $\varphi(\lambda, \lambda t) \neq 0$ that $|\lambda|_\infty = \left|\dfrac{n}{n_0}\right|_\infty$ is bounded; hence, n may assume only finitely many values. So we find:

$$(F\psi)(0, 1) = \int_A \Big(\sum_{\lambda \in Q^*} \varphi(\lambda, \lambda t) \Big)\, dt = \int_A \psi(1, t)\, dt = (B\psi)(0, 1).$$

This completes the proof of Theorem 7.

Now we turn to the proof of Theorem 5. From Theorem 7 and the properties of the operator F established previously it follows immediately that:

1. $B\psi \in L_2(\Omega)$ for every function $\psi \in \Psi_c$;

2. B extends from Ψ_c to the unitary operator $\bar{B} = F$ in $L_2(\Omega)$. Here $\bar{B}$ satisfies the following relation

$$\bar{B}^2 = E$$

where E is the identity operator.

So we have shown that the family Ψ_c has all the properties stated in Theorem 5, which is now proved.

14. The Operator M. Now we proceed to the main task of this section: the investigation of the spectrum of the representation of G_A in the space $L_2(X)$, $X = G_Q \setminus G_A$. For this purpose we make use of the horospherical map of $L_2(X)$:

$$\varphi(y) = \int_{Z_Q \setminus Z_A} f(x_0 z g) \, dz, \tag{1}$$

which associates with every function $f(x) \in L_2(X)$ a function $\varphi(y)$ on Ω that is summable on every compact subset of Ω. In **§ 4.5** we proved that the kernel H^0 of the horospherical map is a closed invariant subspace.

Let H' be the image of $L_2(X)$ under the horospherical map. We endow H' with the structure of a Hilbert space by setting

$$H' \cong L_2(X)/H^0.$$

Since $L_2(X)$ is isomorphic to the direct sum

$$L_2(X) \cong H^0 \oplus H',$$

the description of the spectrum of the representation of G_A in $L_2(X)$ reduces to the same task in H^0 and H'.

The spectrum of the representation in H^0 has already been investigated in **§ 4.6**. We have shown that it is discrete and of finite multiplicity. In this and the following subsections we investigate the spectrum of the representation in H'. The main rôle in this investigation is played by an operator M, which we are about to define.

Let $\psi(y)$ be an arbitrary finite continuous function on Ω. This function gives rise to a functional in the space H':

$$(\psi, \varphi) = \int_\Omega \varphi(y) \overline{\psi(y)} \, dy, \qquad \varphi(y) \in H'. \tag{2}$$

Since the functions $\varphi(y) \in H'$ are summable on every compact subset, the integral (2) converges.

At the end of this subsection we show that (ψ, φ) *is a linear continuous functional relative to* φ *in the space* H'. Hence, it follows by the Riesz theorem that

$$\int_\Omega \varphi(y) \overline{\psi(y)} \, dy = [\varphi, M\psi], \tag{3}$$

where $M\psi \in H'$ and the brackets denote the scalar product in H'.

So we have defined an operator M mapping finite continuous functions on Ω into functions in H'.

We mention the following properties of M:

1. *M commutes with the representation operators $T(g)$, that is,*

$$M[\psi(yg)] = (M\psi)(yg), \qquad g \in G_A \tag{4}$$

for every finite continuous function $\psi(y)$.

This follows immediately from the definition of M.

2. $$(M\psi, \psi) \geq 0$$

for every finite continuous function $\psi(y)$; the parentheses denote the scalar product in $L_2(\Omega)$.

This follows immediately from the equation

$$(M\psi, \psi) = [M\psi, M\psi].$$

3. *The functions $M\psi$, when ψ ranges over the finite continuous functions on Ω, form an everywhere dense subset of H'.*

If we assume the contrary, then there exists a function $f \in H'$, $f \neq 0$, such that $[f, M\psi] = 0$ for every finite continuous function ψ. Hence, by definition of M it follows that

$$\int_\Omega f(y)\overline{\psi(y)}\, dy = 0$$

for every finite continuous function $\psi(y)$. But then $f(y) \equiv 0$, which contradicts the assumption.

Now we prove that (ψ, φ) defined by (2) is, in fact, a continuous linear functional on H'. Obviously this assertion follows immediately from the following lemma.

LEMMA 6. *The following estimate holds for a function $\varphi(y) \in H'$:*

$$\int_K |\varphi(y)|\, dy \leq C_K \, \|\varphi\|_{H'}, \tag{5}$$

where K is an arbitrary compact set in Ω, C_K a constant depending on K only, and $\|\varphi\|_{H'}$ the norm of $\varphi \in H'$.

Proof. We take the collection $\{F\}$ of compact cylindrical sets in X and denote by F_0 the image of the cylindrical set F under the natural map $F \to \Omega$.† By Theorem 1 the sets F_0 form a covering of Ω, therefore every compact set $K \subset \Omega$ can be covered by a finite number of sets F_0. Hence, it suffices to prove the lemma only for the case $K = F_0$.

But for every inverse image $\varphi^*(x)$ of the function $\varphi(y)$ under the horospherical map (1) we have

$$\int_{F_0} |\varphi(y)|\, dy \leq \int_F |\varphi^*(x)|\, dx \leq C_F \, \|\varphi^*\|_{L_2(X)}. \tag{6}$$

† For the definition of cylindrical sets see p. 291.

Since $\varphi^*(x)$ is an arbitrary inverse image, we can take the lower bound on the right side of (6) and so replace $\|\varphi^*\|_{L_2(X)}$ by $\|\varphi\|_{H'}$. Thus,

$$\int_{F_0} |\varphi(y)|\, dy \leq C_F \|\varphi\|_{H'}.$$

This completes the proof of the lemma.

15. An Explicit Expression for M. THEOREM 8. *The following formula holds for the operator M:*

$$(M\psi)(y) = \psi(y) + \int_{Z_A} \psi(y_0 zg)\, dz, \tag{1}$$

where $y_0 = (0, 1)$, and $g \in G_A$ is an arbitrary element of the coset $y = D_Q Z_A g$. In other words,

$$M = E + B, \tag{2}$$

where E is the identity operator and B the operator defined in **§ 4.10.**

To prove (1) we transform the left side of the equation

$$\int_{\Omega} \varphi(y)\psi(y)\, dy = [\varphi, M\psi] \tag{3}$$

into an integral over X. We do this in two steps. First we pass from $\Omega = D_Q Z_A \setminus G_A$ to $D_Q Z_Q \setminus G_A$, and then from $D_Q Z_Q \setminus G_A$ to $X = G_Q \setminus G_A$.

In all the subsequent computations we may assume that the function $\varphi(y)$ is the image of a *continuous* function $f(x) \in L_2(X)$ under the horospherical map and consequently is itself a continuous function. We note that since the continuous functions $f(x) \in L_2(X)$ form an everywhere dense set in $L_2(X)$, their images under the horospherical map form an everywhere dense set in H'.

Thus, first we transform the integral

$$\int_{-z} \varphi(y)\overline{\psi(y)}\, dy \tag{4}$$

into an integral over $D_Q Z_Q \setminus G_A$.

Let $\varphi(g)$ be a function defined on G_A and assuming the constant value $\varphi(y)$ on each coset $y = D_Q Z_A g$. Similarly, we define the function $\psi(g)$. Then we may write:

$$\int_{\Omega} \varphi(y)\overline{\psi(y)}\, dy = \int_{D_Q Z_A \setminus G_A} \varphi(g)\overline{\psi(g)}\, dy. \tag{5}$$

Since $\varphi \in H'$ is the image of a continuous function $f(x) \in L_2(X)$ under the horospherical map,

$$\varphi(g) = \int\limits_{Z_Q \backslash Z_A} f(zg)\, dz.$$

Here $f(g)$ is a continuous function defined on G_A and assuming the constant value $f(x)$ on each coset $x = \Gamma g$.

Since $\psi(g)$ is constant on the cosets of $D_Q Z_A$, we have:

$$\begin{aligned} \int\limits_{D_Q Z_Q \backslash G_A} f(g)\overline{\psi(g)}\, dg &= \int\limits_{D_Q Z_A \backslash G_A} \int\limits_{Z_Q \backslash Z_A} f(zg)\overline{\psi(g)}\, dz\, dy \\ &= \int\limits_{D_Q Z_A \backslash G_A} \varphi(g)\overline{\psi(g)}\, dy \end{aligned} \tag{6}$$

So we find the following equation on the basis of (5) and (6):

$$\int\limits_{\Omega} \varphi(y)\overline{\psi(y)}\, dy = \int\limits_{D_Q Z_Q \backslash G_A} f(g)\overline{\psi(g)}\, dg, \tag{7}$$

where $f(g)$ is a continuous function, the inverse image of $\varphi(y)$ under the horospherical map.

Now we transform the integral on the right side of (7) into an integral over $X = \Gamma \backslash G_A$. We recall that the function $f(g)$ is constant on the cosets of $\Gamma \backslash G_A$.

Let F be a fundamental domain in G_A relative to Γ. Then the sets γF, where $\gamma \in \Gamma$, cover G_A without repetitions. We project all these sets onto $D_Q Z_Q \backslash G_A$. The images of $\gamma_1 F$ and $\gamma_2 F$ either coincide (if $\gamma_2^{-1}\gamma_1 \in D_Q Z_Q$), or are disjoint. We call two elements γ_1 and γ_2 equivalent if the images of $\gamma_1 F$ and $\gamma_2 F$ coincide. Then we can write:

$$\begin{aligned} \int\limits_{D_Q Z_Q \backslash G_A} f(g)\overline{\psi(g)}\, dg &= \sum_{\gamma} \int\limits_{\gamma F} f(g)\overline{\psi(g)}\, dg \\ &= \sum_{\gamma}{}' \int\limits_{F} f(g)\overline{\psi(\gamma g)}\, dg, \end{aligned} \tag{8}$$

where the summation is taken over the pairwise inequivalent γ.

Since $\psi(g)$ is finite on $D_Q Z_Q \backslash G_A$, its support intersects only finitely many pairwise inequivalent sets γF. Consequently, the summation in (8) is, in fact, over finitely many elements γ. Hence, summation and integration in (8) may be interchanged. As a result, on the basis of (7) and (8), we find the following equation:

$$\int\limits_{\Omega} \varphi(y)\overline{\psi(y)}\, dy = \int\limits_{F} f(g)\left(\sum_{\gamma}{}' \overline{\psi(\gamma g)}\right) dg. \tag{9}$$

Observe that $\sum_{\gamma}' \psi(\gamma g)$ does not depend on the choice of the element γ among equivalent ones (because $\psi(g)$ is constant on the cosets of $D_Q Z_Q \setminus G_A$) and does not change when g is replaced by $\gamma_0 g$, $\gamma_0 \in \Gamma$. Consequently, the function $\sum_{\gamma}' \psi(\gamma g)$ is constant on the cosets of $\Gamma \setminus G_A$, and so can be regarded as a function on $X = \Gamma \setminus G_A$, as is $f(g)$. So we have:

$$\int_{\Omega} \varphi(y)\overline{\psi(y)}\, dy = \int_{X} f(x)\overline{\psi_1(x)}\, dx \tag{10}$$

where

$$\psi_1(x) = \sum_{\gamma}' \psi(\gamma g). \tag{11}$$

We show that $\psi_1(x)$ is orthogonal to H^0. For let $f(x)$ be an arbitrary continuous function in H^0 and $\varphi(y)$ its image under the horospherical map. Then, by definition of H^0, we have $\varphi(y) = 0$, hence by (10)

$$\int_{X} f(x)\overline{\psi_1(x)}\, dx = 0,$$

that is, $\psi_1(x)$ is orthogonal to every continuous function $f \in H^0$. Since the continuous functions $f \in H^0$ form an everywhere dense set in H^0, $\psi_1(x)$ is orthogonal to H^0.

From the fact that $\bar{\psi}_1$ is orthogonal to H^0 it follows that

$$\int_{X} f(x)\overline{\psi_1(x)}\, dx = [\varphi, \hat{\psi}_1], \tag{12}$$

where $\hat{\psi}_1$ is the image of $\psi_1(x)$ under the horospherical map.

Comparing (10) and (12) with the original equation (3) we find that

$$[\varphi, M\psi] = [\varphi, \hat{\psi}_1]$$

for the everywhere dense set of functions $\varphi \in H'$. Consequently,

$$M\psi = \hat{\psi}_1.$$

So we have established that the application of the operator M to $\psi(y)$ reduces to the following operations. First we construct from $\psi(y)$ a function on $X = \Gamma \setminus G_A$:

$$\psi_1(x) = \sum_{\gamma}' \psi(\gamma g).$$

Then we apply the horospherical map to the function $\psi_1(x)$. Thus, the operator M is given by the following formula:

$$(M\psi)(y) = \int_{Z_Q \setminus Z_A} \Big(\sum_{\gamma \in D_Q Z_Q \setminus \Gamma}' \psi(\gamma z g) \Big)\, dz; \tag{13}$$

In $\sum'$ precisely one representative is taken from every coset of $D_Q Z_Q \setminus \Gamma$; g is an arbitrary element of the coset of $D_Q Z_A \setminus G_A$ corresponding to y.

We transform formula (13). For this purpose we single out in the sum $\sum'$ the term corresponding to the unit coset $\psi(zg) = \psi(g)$. It is easy to verify that each of the remaining cosets of $D_Q Z_Q \gamma$ contains one and only one representative† of the form sz', where $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $z' \in Z_Q$. So we find:

$$(M\psi)(y) = \psi(y) + \int\limits_{Z_Q \setminus Z_A} \left(\sum_{z' \in Z} \psi(y_0 z' z g) \right) dz,$$

where $y_0 \in \Omega$ is the element corresponding to the coset $D_Q Z_A s$, that is, $y_0 = (0, 1)$.

We recall that the summation is, in fact, over a finite set of elements. Therefore, it is clear that the summation over Z_Q and the integration over $Z_Q \setminus Z_A$ reduces to integration over the whole of Z_A. So we have finally:

$$(M\psi)(y) = \psi(y) + \int\limits_{Z_A} \psi(y_0 z g)\, dz,$$

as required.

16. The Family $\mathscr{M}$ of Functions on Ω. The object of this subsection is to prove the following theorem.

THEOREM 9. *There exists a family $\mathscr{M}$ of finite continuous functions defined on Ω and having the following properties:*

1. *$\mathscr{M}$ is everywhere dense in $L_2(\Omega)$.*
2. *$M\varphi \in L_2(\Omega)$ for every $\varphi \in \mathscr{M}$.*
3. *Let H' be the image of $L_2(\Omega)$ under the horospherical map, and $H'' \subset H'$ be the subspace of H' orthogonal to the space C of constants. Then the functions $M\varphi$, $\varphi \in \mathscr{M}$ belong to H', and form in H'' an everywhere dense set.*

We proceed to construct such a family $\mathscr{M}$. As a preliminary we recall that the space $E \subset Y$ introduced in **§ 4.6** is a topological product

$$E = V_\infty \times V_2 \times \cdots \times V_p \times \cdots,$$

where V_∞ is the real affine plane with the point zero removed, and V_p the domain consisting of the points $y_p \in Q_p^2$ of norm 1.

† This follows from the fact that every rational matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in which $b \neq 0$ can be represented in one and only one way in the form $\gamma = z_1 \delta s z_2$, where $\delta \in D_Q$, $z_1, z_2 \in Z_Q$.

We introduce the set $\mathscr{M}_0$ of functions $\varphi(y)$ defined on E and of the form

$$\varphi(y) = \varphi_\infty(y_\infty)\varphi_2(y_2)\cdots\varphi_p(y_p)\cdots, \tag{1}$$

where the factors satisfy the following conditions:

1. $\varphi_\infty(y_\infty)$ is an infinitely differentiable finite function on V_∞, that is, $\varphi_\infty(y_\infty) = 0$ for sufficiently small and for sufficiently large values of $|y_\infty|$, such that

$$\int_{V_\infty} \varphi_\infty(y_\infty)\, dy_\infty = 0; \tag{2}$$

2. For every $p = 2, 3, \ldots$ the function $\varphi_p(y_p)$ on V_p has the following form:

$$\varphi_p(y_p) = \begin{cases} 1, & \text{when } |y_p - y_p^0| \leq p^{-m_p}, \\ 0, & \text{when } |y_p - y_p^0| > p^{-m_p}, \end{cases} \tag{3}$$

where y_p^0 is a point on V_p, and m_p a nonnegative integer. Also $m_p = 0$ for all but a finite number of p.

We denote by $\mathscr{M}$ the set of functions

$$\psi(y) = \varphi(y) + \varphi(-y), \qquad \varphi \in \mathscr{M}_0 \tag{4}$$

and all their finite linear combinations. Obviously the functions $\psi(y)$ are finite continuous functions on Ω.

We shall show that the set $\mathscr{M}$ has all the properties stated in Theorem 9. We divide the proof into several stages.

LEMMA 7. *The set $\mathscr{M}$ is everywhere dense in $L_2(\Omega)$.*

Proof. It is sufficient to show that the finite linear combinations of functions $\varphi \in \mathscr{M}_0$ form an everywhere dense set in $L_2(E)$. But this follows easily from the fact that the set of functions $\varphi_\infty(y_\infty)$ satisfying condition 1 is everywhere dense in $L_2(V_\infty)$, and the set of functions $\varphi_p(y_p)$ of the form (3) is everywhere dense in $L_2(V_p)$, $p = 2, 3, \ldots$.

LEMMA 8. *Let $f(y) \in H'$, where H' is the image of $L_2(X)$, $X = \Gamma \setminus G_A$, under the horospherical map. If*

$$\int_\Omega f(y)\overline{\psi(y)}\, dy = 0 \tag{5}$$

for every function $\psi \in \mathscr{M}$, then $f(y) = \text{const}$.

Proof. First we observe that the condition (5) is equivalent to the following:

$$\int_E f(y)\overline{\varphi(y)}\, dy = 0 \tag{6}$$

for every function $\varphi(y) \in \mathscr{M}_0$. Now we prove the following auxiliary proposition:

Proposition 1. *If the function $f(y)$, $y \in E$, is summable on every compact set in E and satisfies the condition* (6), *then it does not depend on y_∞.*

Suppose the contrary. Then we can find a compact open subset $V^{(1)} \subset V$, where $V = V_2 \times \cdots \times V_p \times \cdots$, and where $V^{(1)}$ is defined by the conditions

$$|y_p - y_p^0| \leq p^{-m_p}, \qquad p = 2, 3, \ldots,$$

and is such that the function

$$f_\infty(y_\infty) = \int_{V^{(1)}} f(y_\infty, v)\, dv$$

is not constant.

We introduce the set $\mathscr{M}_\infty$ of finite infinitely differentiable functions $\varphi_\infty(y_\infty)$ defined on V_∞ and satisfying (2), and we show that

$$\int_{V_\infty} f_\infty(y_\infty)\overline{\varphi_\infty(y_\infty)}\, dy_\infty = 0 \tag{7}$$

for every function $\varphi_\infty \in \mathscr{M}_\infty$.

For we set

$$\varphi(y) \equiv \varphi(y_\infty, v) = \begin{cases} \varphi_\infty(y_\infty), & \text{when } v \in V^{(1)}, \\ 0 \quad , & \text{when } v \notin V^{(1)}, \end{cases}$$

where $\varphi_\infty(y_\infty) \in \mathscr{M}_\infty$. Obviously, $\varphi(y) \in \mathscr{M}_0$. Substituting this function in (6) we obtain the relation (7).

Let us show that (7) *implies* that $f_\infty(y_\infty) = \text{const}$. The contradiction *implies* Proposition 1.

For this purpose we go over to polar coordinates on V_∞. Let

$$F_k(\tau) = \int_0^{2\pi} f_\infty(\tau\cos\theta, \tau\sin\theta)e^{ik\theta}\, d\theta.$$

Note that the set $\mathscr{M}_\infty$ necessarily contains the functions of the form

$$\varphi_\infty(\tau\cos\theta, \tau\sin\theta) = \tau^{-1}\psi(\tau)e^{-ik\theta}, \quad k \neq 0,$$

and

$$\varphi_\infty(\tau\cos\theta, \tau\sin\theta) = \tau^{-1}\frac{d\psi(\tau)}{d\tau},$$

where $\psi(\tau)$ is an infinitely differentiable function defined on the half-line $0 \leq \tau < \infty$ and vanishing for sufficiently small and sufficiently large values of τ. Substituting these functions in (7) we find:

$$\int_0^\infty F_0(\tau)\frac{d\overline{\psi(\tau)}}{d\tau}\, d\tau = 0; \qquad \int_0^\infty F_k(\tau)\overline{\psi(\tau)}\, d\tau = 0, \quad k \neq 0.$$

Since these equations are valid for every finite function $\psi(\tau)$, it follows from them that $F_0(\tau) = C_0$, $F_k(\tau) = 0$ for $k \neq 0$. Consequently, $f_\infty(y_\infty) = \text{const.}$, which contradicts our assumption. And so Proposition 1 is proved.

Now we proceed to the proof of Lemma 7.

Suppose that $f(y)$, $y \in E$, satisfies the conditions of the lemma. We consider the inverse image $F(x) \in L_2(X)$ of $f(y)$ in the orthogonal complement to H^0. Here H^0 is the kernel of the horospherical map.

We show that $F(x) = \text{const.}$; the assertion of the lemma that $f(y) = \text{const.}$ follows from this.

Suppose the contrary: that $F(x) \neq \text{const.}$ Without loss of generality we may assume that $F(x)$ is a continuous function; otherwise we could go over from $F(x)$ to the function

$$F_1(x) = \int_{G_A} F(xg)u(g)\, dg,$$

where $u(g)$ is a smooth finite function on G_A. It is easy to check that $F_1(x)$ also lies in the orthogonal complement to H^0 and that its image satisfies the conditions of the lemma.

Since the function $f(y)$ does not depend on y_∞, it is invariant under the operators

$$T_\infty(g_\infty)f(y) = f(y\tilde{g}_\infty) \quad \text{where } \tilde{g}_\infty = (g_\infty, 1, \ldots, 1, \ldots).$$

Consequently, the function $F(x)$ has the same property. Thus, $F(x\tilde{g}_\infty) = F(x)$.

Besides, the set of elements $x_0\tilde{g}_\infty$ is everywhere dense in X; this fact can be established by simple modifications of the arguments in **§ 4.2** (see the Remark on p. 353). Thus, from the equation $F(x\tilde{g}_\infty) = F(x)$ and the continuity of $F(x)$ it follows that $F(x) = \text{const.}$ and the lemma is proved.

LEMMA 9. *For every function $\psi \in \mathcal{M}$ we have $M\psi \in H''$, where $H'' \subset H'$ is the orthogonal complement in H' to the subspace C of constants. Also, the functions $M\psi$, where $\psi \in \mathcal{M}$, form an everywhere dense set in H''.*

Proof. By the definition of the operator M we have

$$[f, M\psi] = \int_\Omega f(y)\overline{\psi(y)}\, dy,$$

for every function $f \in H'$ (the brackets denote the scalar product in H').

We show that $M\psi \in H'$, that is, $M\psi$ is orthogonal to the constants. For if $f(y) = c$, then

$$[f, M\psi] = c\int_\Omega \overline{\psi(y)}\, dy.$$

But the functions $\psi(y) \in \mathscr{M}$ satisfy the relation $\int\limits_{\Omega} \psi(y)\, dy = 0$. Consequently, $[f, M\psi] = 0$.

Conversely, suppose that a function $f \in H'$ is orthogonal to all $M\psi$, $\psi \in \mathscr{M}$, that is, $[f, M\psi] = 0$. Then we have

$$\int\limits_{\Omega} f(y)\overline{\psi(y)}\, dy = 0 \quad \text{for every } \psi \in \mathscr{M}.$$

Hence it follows by Lemma 8 that $f(y) = \text{const}$. Consequently, the functions $M\psi$, $\psi \in \mathscr{M}$, are everywhere dense in H'', and the lemma is proved.

LEMMA 10. *If $\psi \in \mathscr{M}$, then $M\psi \in L_2(\Omega)$.*

Proof. Since $M = E + B$, where E is the unit operator, the assertion of the lemma is equivalent to the fact that

$$B\psi \in L_2(\Omega).$$

We recall that $\mathscr{M}$ is the set of finite linear combinations of functions of the form:

$$\psi(y) = \varphi(y) + \varphi(-y), \tag{8}$$

where

$$\varphi(y) = \varphi_\infty(y_\infty)\,\varphi_2(y_2) \cdots \varphi_p(y_p) \cdots.$$

Here $\varphi_\infty(y_\infty)$ and $\varphi_p(y_p)$, $p = 2, 3, \ldots$, satisfy the following conditions:

$$\int \varphi_\infty(y_\infty)\, dy_\infty = 0.$$

$$\varphi_p(y_p) = \begin{cases} 1, & \text{when } |y_p - y_p^0| \le p^{-m_p}, \\ 0, & \text{when } |y_p - y_p^0| > p^{-m_p}. \end{cases} \tag{9}$$

Therefore, it is enough to prove the assertion for functions $\psi(y)$ of the indicated form.

We apply the formula (7) in **§ 4.10** for B to a function $\psi(y)$ of the form (8) and obtain:

$$(B\psi)(a_\tau \circ v) = \frac{1}{\tau} \sum_{n \ne 0} \left\{ \int\limits_{-\infty}^{+\infty} \varphi_\infty\left(t_\infty, \frac{n}{\tau}; v_\infty\right) dt_\infty \prod_p \int\limits_{\max(|n|_p, |t_p|) = 1} \varphi_p(t_p, n; v_p)\, dt_p \right\}. \tag{10}$$

The summation is taken over the integers $n \ne 0$. Here, as in **§ 4.10**, we use the following notation:

$a_\tau = (\tilde{y}_\infty, \tilde{y}_2, \ldots, \tilde{y}_p, \ldots)$, where $\tilde{y}_\infty = (\tau, 0)$, $\tau = |y|$, $\tilde{y}_p = (1, 0)$, $p = 2, 3, \ldots$; $v = (v_\infty, v_2, \ldots, v_p, \ldots)$, where

$$|v_\infty| = |v_2| = \cdots = |v_p| = \cdots = 1;$$

$\varphi_\infty(y_\infty; v_\infty) = \varphi_\infty(y_\infty \circ \bar{v}_\infty^{-1})$, $\varphi_p(y_p; v_p) = \varphi_p(y_p \circ \bar{v}_p^{-1})$; the sign $\circ$ denotes the operation of multiplication in E.

We compute the integrals

$$\int_{\max(|n|_p, |t_p|)=1} \varphi_p(t_p, n; v_p)\, dt_p, \tag{11}$$

by using the explicit expression (9) for φ_p. We set†

$$q = \prod_p p^{m_p}$$

and treat separately the case $p \times q$, that is, $m_p = 0$, and the case $p \mid q$, that is, $m_p > 0$. In the first case, $\varphi_p(y_p) \equiv 1$ on the set V_p of elements y_p of norm 1. Therefore, the integral (11) is equal to the measure of the set of elements $t_p \in Q_p$ for which $\max(|n|_p, |t_p|) = 1$. So we have: if $p \nmid q$, then

$$\int_{\max(|n|_p, |t_p|)=1} \varphi_p(t_p, n; v_p)\, dt_p = \begin{cases} 1, & \text{when } p \nmid n, \\ 1 - \dfrac{1}{p}, & \text{when } p \mid n. \end{cases}$$

Now we consider the case $p \mid q$. It is easy to check that if $p \mid q$, that is, $m_p > 0$, then

$$\int_{\max(|n|_p, |t_p|)=1} \varphi_p(t_p, n; v_p)\, dt_p = \begin{cases} p^{-m_p}, & \text{when } |n - z_p^2| \le p^{-m_p}, \\ 0, & \text{when } |n - z_p^2| > p^{-m_p}. \end{cases}$$

Here z_p^2 denotes the second coordinate of the point $z_p = y_p^0 \circ \bar{v}_p$.

By definition

$$\varphi_p(t_p, n; v_p) = \begin{cases} 1, & \text{when } |t_p - z_p^1| \le p^{-m_p}, \quad |n - z_p^2| \le p^{-m_p}, \\ 0, & \text{when } |t_p - z_p^1| > p^{-m_p} \quad \text{or,} \quad |n - z_p^2| > p^{-m_p}, \end{cases}$$

where z_p^1, z_p^2 are the coordinates of the point $z_p = y_p^0 \circ \bar{v}_p$. Hence, the integral is zero when $|n - z_p^2| > p^{-m}p$. But if $|n - z_p^2| \le p^{-m_p}$, then the integral is equal to the measure of the set of elements $t_p \in Q_p$ for which $|t_p - z_p^1| \le p^{-m_p}$ and $\max(|n|_p, |t_p|) = 1$. Next we show that this measure is equal to p^{-m_p}. Again we treat separately the case $p \nmid n$ and $p \mid n$. In the first case, the set is given by the conditions $|t_p - z_p^1| \le p^{-m_p}$, $|t_p| \le 1$, and therefore, has the measure p^{-m_p}. In the second case, the set is given by the conditions $|t_p| = 1$, $|t_p - z_p^1| \le p^{-m_p}$. Note that $|z_p^1| = 1$. This follows from the conditions:

$$|n|_p < 1, \qquad |n - z_p^2| < 1, \qquad \max(|z_p^1|, |z_p^2|) = 1).$$

Consequently, in the second case the measure of the set is also p^{-m_p}.

Observe that the integers $n \ne 0$ satisfying the system of inequalities

$$|n - z_p^2| \le p^{-m_p}, \qquad p = 2, 3, \ldots,$$

† The expression makes sense, because $m_p = 0$ for all but a finite number of p.

form a residue class modulo $q = \prod_p p^{m_p}$. In other words, they range over all the integers of the form

$$n = a + qm, \qquad n \neq 0,$$

where a is a fixed integer from the interval $0 \leq a < q$, and $m = 0, \pm 1, \pm 2, \ldots$. So we find:

$$\prod_p \int_{\max(|n|_p, |t_p|)=1} \varphi_p(t_p, n; v_p)\, dt_p = \begin{cases} q^{-1} \prod_{\substack{p|n, \\ p \nmid q}} \left(1 - \frac{1}{p}\right), & \text{when } n \equiv a (\text{mod } q), \\ 0, & \text{when } n \not\equiv a\ (\text{mod } q). \end{cases} \tag{12}$$

To simplify (12) we introduce the numerical character $\chi_q(n)$ defined by the conditions:

$$\chi_q(n) = \begin{cases} 1, & \text{when } (n, q) = 1, \\ 0, & \text{when } (n, q) > 1. \end{cases} \tag{13}$$

Next we define a multiplicative function $\psi_q(n)$ by the formula

$$\psi_q(n) = \prod_{p|n} \left(1 - \frac{\chi_q(p)}{p}\right); \qquad \psi_q(\pm 1) = 1. \tag{14}$$

In this notation we have:

$$q^{-1} \prod_{\substack{p|n, \\ p \nmid q}} \left(1 - \frac{1}{p}\right) = q^{-1} \psi_q(n).$$

As a result, the formula (10) for $(B\psi)(a_\tau \circ v)$ assumes the following form:

$$(B\psi)(a_\tau \circ v) = q^{-1} \tau^{-1} \sum_{n \in g} \psi_q(n) f\left(\frac{n}{\tau}; v_\infty\right), \tag{15}$$

where

$$f(s; v_\infty) = \int_{-\infty}^{+\infty} \varphi_\infty(t, s; v_\infty)\, dt. \tag{16}$$

The summation is taken over the set g of integers $n \neq 0$ of the form $n = a + qm$, $m = 0, \pm 1, \pm 2, \ldots$ (a is a fixed integer, $0 \leq a < q$).†

Now we establish properties of the function $f(s; v_\infty)$. From the conditions imposed on $\varphi_\infty(y_\infty)$ it follows immediately that: $f(s; v_\infty)$

† The number a depends, of course, on $(v_2, v_3, \ldots, v_p, \ldots)$. However, it is easy to verify that a is a piecewise constant function of $(v_2, v_3, \ldots, v_p, \ldots)$.

is a finite, infinitely differentiable function of s and satisfies the following condition

$$\int_{-\infty}^{+\infty} f(s; v_\infty)\, ds = 0. \tag{17}$$

Under these conditions on $f(s; v_\infty)$ it can be shown that

$$\int_0^\infty \left|\tau^{-1} \sum_{n\in g} \psi_q(n) f\left(\frac{n}{\tau}; v_\infty\right)\right|^2 \tau\, d\tau < C, \tag{18}$$

uniformly in v. The proof of the inequality (18) will be given separately in **§4 Appendix 1.2.**

Since

$$\begin{aligned}\|(B\psi)(y)\|^2 &\equiv \int |(B\psi)(a_\tau \circ v)|^2\, \tau\, d\tau\, dv \\ &= q^{-2}\int \left|\tau^{-1} \sum_{n\in g} \psi_q(n) f\left(\frac{n}{\tau}; v_\infty\right)\right|^2 \tau\, d\tau\, dv,\end{aligned}$$

we find by (18) that $\|B\psi\|^2 < \infty$, and the lemma is proved.

On the basis of Lemmas 7, 9 and 10 we conclude that $\mathscr{M}$ has all the properties stated in Theorem 9, and the proof of Theorem 9 is now complete.

17. Decomposition of the Representation in H' into Irreducible Representations. THEOREM 10. (THE MAIN THEOREM). *The image H' of $L_2(X)$ under the horospherical map splits into the direct sum*

$$H' = C \oplus H'' \tag{1}$$

of the one-dimensional space of constants C and the subspace H'' of $L_2(\Omega)$. The representation of G_A in H'' splits into the same irreducible representations as in $L_2(\Omega)$; however, in contrast to the latter, it contains each irreducible representation only with multiplicity 1.

Proof. In **§4.13** we defined a unitary operator $\bar{B}$ in $L_2(\Omega)$ such that

$$\bar{B}^2 = E, \tag{2}$$

where E is the unit operator. We recall that $\bar{B}$ is an extension of B from a certain subset of functions everywhere dense in $L_2(\Omega)$.

In $L_2(\Omega)$ we introduce a bounded operator $\bar{M}$ by the following formula:

$$\bar{M} = E + \bar{B}. \tag{3}$$

Note that $\bar{M}$ is self-adjoint and satisfies by virtue of (2) the following relation:

$$\bar{M}^2 = 2\bar{M}. \tag{4}$$

Let H'' be the orthogonal complement in H' to the subspace of constants C. We show that

$$H'' = \bar{M}(L_2(\Omega)), \tag{5}$$

that is, H'' coincides with the image of $L_2(\Omega)$ under $\bar{M}$. Hence it follows, in particular, that $H'' \subset L_2(\Omega)$.

For the proof we take the set $\mathscr{M}$ of finite continuous functions defined on Ω and satisfying the conditions of Theorem 9. We show that

$$\bar{M}\varphi = M\varphi \qquad \text{for every function } \varphi \in \mathscr{M}.$$

Here M is the operator introduced in **§ 4.14**. For if $\varphi \in \mathscr{M}$, then $B\varphi \in L_2(\Omega)$; therefore, by the Corollary to Theorem 5 we have $B\varphi = \bar{B}\varphi$. But, according to Theorem 8,

$$M = E + B$$

consequently, $M\varphi = \bar{M}\varphi$.

We consider the image $\bar{M}(\mathscr{M})$ of $\mathscr{M}$ under $\bar{M}$. Since $\mathscr{M}$ is everywhere dense in $L_2(\Omega)$, its image $\bar{M}(\mathscr{M})$ is an everywhere dense subset of $\bar{M}(L_2(\Omega)) \subset L_2(\Omega)$.

On the other hand, the set $\bar{M}(\mathscr{M}) = M(\mathscr{M})$ is contained as an everywhere dense subset in H'' (Property 3 of $\mathscr{M}$, see Theorem 9.)

To prove (5) it is enough to check that the completions of $\bar{M}(\mathscr{M})$ in the norm of $L_2(\Omega)$ and in the norm of H'' are the same. But this follows immediately from the next easily verifiable equation:

$$(\bar{M}_\varphi, \bar{M}_\psi) = 2[\bar{M}_\varphi, \bar{M}_\psi] \qquad \text{for any } \varphi, \psi \in \mathscr{M}. \tag{6}$$

Here the parentheses denote the scalar product in $L_2(\Omega)$, and the brackets the scalar product in H'.

Proof of (6).

$$(\bar{M}_\varphi, \bar{M}_\psi) = (\bar{M}^2\varphi, \psi) = 2(\bar{M}\varphi, \psi) = 2[\bar{M}_\varphi, \bar{M}_\psi].$$

Note that H'' may be characterized as the eigenspace for the eigenvalue 1 of $\bar{B}$. This follows immediately from the equations:

$$H'' = (E + \bar{B})L_2(\Omega) \quad \text{and} \quad \bar{B}^2 = E.$$

Finally, we show that H'' has a spectrum of multiplicity 1. From **§ 4.9** we know the representation in $L_2(\Omega)$ splits into irreducible representations acting in the spaces H_π of homogeneous functions. Here the representations in the spaces H_π and H_π^-, and only they, are equivalent. Thus, the irreducible representations occur in $L_2(\Omega)$ with multiplicity 2.

The operator $\bar{B}$, since it commutes with the representation operators, carries the sum of equivalent spaces

$$H_\pi + H_{\pi^{-1}}$$

into itself. From the formula for the operator F of Fourier transformation in $L_2(\Omega)$, which as we know coincides with $\bar{B}$ (see **§ 4.13**), it follows that

$$\bar{B}H_\pi = H_{\pi^{-1}},$$

that is, $\bar{B}$ interchanges the spaces H_π and $H_{\pi^{-1}}$.

Hence, it follows that the operator $\bar{M} = E + \bar{B}$ carries the space $H_\pi + H_{\pi^{-1}}$ into the subspace H'' of functions of the form $\psi_\pi + \bar{B}\psi_\pi$, $\psi_\pi \in H_\pi$. This shows that the representation in the subspace $H'' = \bar{M}(L_2(\Omega))$ has a spectrum of multiplicity 1, and the theorem is proved.

18. Connection of the Operator of the Horospherical Automorphism B with Dirichlet L-Functions. In this subsection we construct a system of functions $\varphi_\pi \in H_\pi$ for which we can find an explicit expression for the operator B. The formulae we obtain are interesting in that they establish a link between the degeneracy of the operator $M = E + B$ in the spaces $H_\pi + H_{\pi^{-1}}$ and the functional equation for the Dirichlet L-functions.

First of all, we recall the description of all characters $\pi(\lambda)$ on the group of ideles that are identically equal to 1 on the subgroup Q^* of principal ideles.

Let

$$\pi = (\pi_\infty, \pi_2, \ldots, \pi_p, \ldots)$$

be a character on the group of ideles

$$\lambda = (\lambda_\infty, \lambda_2, \ldots, \lambda_p, \ldots),$$

that is,

$$\pi(\lambda) = \pi_\infty(\lambda_\infty)\pi_2(\lambda_2) \cdots \pi_p(\lambda_p) \cdots$$

The characters occurring in this product may be written in the following form:

$$\pi_\infty(\lambda_\infty) = |\lambda_\infty|_\infty^s \operatorname{sign}^\nu \lambda_\infty; \tag{1}$$

$$\pi_p(\lambda_p) = |\lambda_p|_p^{s_p}\, \theta_p(\lambda_p), \qquad p = 2, 3, \ldots \tag{1'}$$

Here s and s_p are arbitrary complex numbers; $\nu = 0, 1$; $\theta_p(\lambda_p)$ is a unitary character for which

$$\theta_p(p) = 1.$$

Furthermore, only finitely many characters $\theta_p(\lambda_p)$ are not identically equal to 1.

The condition that the character $\pi(\lambda)$ is identically equal to 1 on the subgroup of principal ideles λ is obviously equivalent to the following relations:

$$\pi_\infty(-1)\pi_2(-1) \ldots \pi_q(-1) \ldots = 1$$

and

$$\pi_\infty(p)\pi_2(p) \ldots \pi_q(p) \ldots = 1$$

for every prime number p. When we substitute here the expressions for $\pi_\infty(x)$, $\pi_2(x)$, $\ldots$, $\pi_p(x)$, $\ldots$, we find

$$\begin{aligned} &\theta_2(-1)\cdots\theta_q(-1)\cdots = \operatorname{sign}^\nu(-1); \\ &p^{s-s_p}\theta_2(p)\cdots\theta_p(p)\cdots = 1. \end{aligned} \tag{2}$$

Let $\theta(\lambda)$ be the unitary character on the multiplicative group of rational numbers that is defined by the formula

$$\theta(\lambda) = \theta_2(\lambda)\cdots\theta_p(\lambda)\cdots, \tag{3}$$

Then (2) can be written in the form

$$\theta(-1) = \operatorname{sign}^\nu(-1), \qquad p^{s_p} = p^s\theta(p).$$

Thus, *every character*

$$\pi = (\pi_\infty, \pi_2, \ldots, \pi_p, \ldots)$$

on the group of ideles that is identically equal to 1 on the subgroup of principal ideles has the following form:

$$\pi_\infty(\lambda_\infty) = |x_\infty|_\infty^s \operatorname{sign}^\nu \lambda_\infty, \tag{4}$$

$$\pi_p(\lambda_p) = |x_p|^{s_p}\,\theta_p(\lambda_p), \qquad p = 2, 3, \ldots, \tag{5}$$

where $\theta_p(\lambda_p)$ are arbitrary characters such that $\theta_p(p) = 1$, s is a complex number, and the exponents ν and s_p are determined by the formulae

$$\operatorname{sign}^\nu(-1) = \theta(-1), \quad p^{s_p} = p^s\theta(p),$$

where

$$\theta(\lambda) = \theta_2(\lambda)\cdots\theta_p(\lambda), \ldots, \ \lambda \in Q^*.$$

Henceforth, only such characters $\pi(\lambda)$ are considered. We extend each of the characters $\pi_p(\lambda_p)$ defined by (4) and (5) to a character on a quadratic extension of the field. Specifically, we extend the character

$$\pi_\infty(x_\infty) = |x_\infty|_\infty^s \operatorname{sign}_\nu x_\infty,$$

which is defined on the multiplicative group of real numbers, to a character on the group of complex numbers according to the following formula:

$$\pi_\infty(z) = |z|^s e^{\nu \arg z}.$$

Next, we extend the character

$$\pi_p(x_p) = |x_p|_p^{s_p}\,\theta_p(x_p),$$

which is defined on the multiplicative group of the field of p-adic numbers Q_p, arbitrarily to a character on the quadratic $Q_p(\sqrt{\varepsilon_p})$ of Q_p, where ε_p is an element of the field of norm 1 (and not a square in Q_p). We denote the character so obtained by the same letter

π_p. The points of the underlying affine space are denoted as follows:

$$a = ((x, y), (x_2, y_2), \ldots, (x_p, y_p), \ldots).$$

We introduce functions of the following form

$$\varphi_\pi^{(1)}(a) = \pi_\infty(x + iy)\,|x + iy|^{-1} \prod_p \pi_p(x_p + \sqrt{\varepsilon_p}y_p)\,|x_p + \sqrt{\varepsilon_p}y_p|^{-1}. \tag{6}$$

$$\varphi_\pi^{(2)}(a) = \pi_\infty(x - iy)\,|x - iy|^{-1} \prod_p \pi_p(x_p - \sqrt{\varepsilon_p}y_p)\,|x_p - \sqrt{\varepsilon_p}y_p|^{-1}. \tag{7}$$

Our object is to compute

$$B\varphi_\pi^{(1)}(a) \quad \text{and} \quad B\varphi_\pi^{(2)}(a),$$

where B is the operator of the horospherical automorphism.

Obviously,

$$B\varphi_\pi^{(1)}(a) = B_\infty(\pi_\infty(x + iy)\,|x + iy|^{-1}) \prod_p B_p(\pi_p(x_p + \sqrt{\varepsilon_p}y_p)\,|x_p + \sqrt{\varepsilon_p}y_p|^{-1}), \tag{8}$$

$$B\varphi_\pi^{(2)}(a) = B_\infty(\pi_\infty(x - iy)\,|x - iy|^{-1}) \prod_p B_p(\pi_p(x_p - \sqrt{\varepsilon}y_p)\,|x_p - \sqrt{\varepsilon}y_p|), \tag{9}$$

where B_p is the operator of the horospherical map corresponding to the group of matrices over the field of p-adic numbers.

Formulae for the operators B_p were obtained in Chapter 2, **§ 3.11**:

$$B_\infty(\pi_\infty(x + iy)\,|x + iy|^{-1}) = \lambda_\infty^{(1)}(\pi_\infty)\pi_\infty^{-1}(x - iy)\,|x - iy|^{-1},$$

where

$$\lambda_\infty^{(1)}(\pi_\infty) = \begin{cases} \mathrm{B}\left(-\dfrac{s}{2}, \dfrac{1}{2}\right), & \text{when } \nu = 0 \\ i\mathrm{B}\left(-\dfrac{s-1}{2}, \dfrac{1}{2}\right), & \text{when } \nu = 1 \end{cases}$$

($\mathrm{B}(x, y)$ is the classical Beta-function);

$$B_p(\pi_p(x_p + \sqrt{\varepsilon_p}y_p)\,|x_p + iy_p|^{-1}) = \lambda_p^{(1)}(\pi_p)\pi_p^{-1}(x_p - \sqrt{\varepsilon_p}y_p)\,|x_p - iy_p|^{-1},$$

where

$$\lambda_p^{(1)}(\pi_p) = \begin{cases} \dfrac{1 - p^{s_p - 1}}{1 - p^{s_p}}, & \text{when } \theta_p(\lambda_p) \equiv 1, \\ p^{-n/2}\mu_p^{(1)}(\pi_p), & \text{when } \theta_p(\lambda_p) \not\equiv 1. \end{cases}$$

Here n is the rank of θ_p, that is, the least natural number n for which $\theta_p(1 + ps) \equiv 1$ for $|s| \leq 1$; $|\mu_p(\pi_p)| = 1$.

On the basis of these formulae we find that

$$B\varphi_{\pi}^{(1)} = \lambda^{(1)}(\pi)\,\varphi_{\pi^{-1}}^{(2)}, \tag{10}$$

where

$$\lambda^{(1)}(\pi) = \lambda_{\infty}^{(1)}(\pi_{\infty})\,\lambda_2^{(1)}(\pi_2)\cdots\lambda_p^{(1)}(\pi_p)\cdots$$

Similarly, on the basis of the formulae of Chapter 2 we find that

$$B\varphi_{\pi}^{(2)} = \lambda^{(2)}(\pi)\,\varphi_{\pi^{-1}}^{(1)}, \tag{11}$$

where

$$\lambda^{(2)}(\pi) = \lambda_{\infty}^{(2)}(\pi_{\infty})\,\lambda_2^{(2)}(\pi_2)\cdots\lambda_p^{(2)}(\pi_p)\cdots,$$

and the factors $\lambda_p^{(2)}(\pi_p)$ are connected with $\lambda_p^{(1)}(\pi_p$ by the following relations:

$$\lambda_{\infty}^{(2)}(\pi_{\infty}) = (-1)^{\nu}\lambda_{\infty}^{(1)}(\pi_{\infty});$$

$$\lambda_p^{(1)}(\pi_p)\,\lambda_p^{(2)}(\pi_p^{-1}) = p^{-n}\pi_p(-1),$$

where n is the rank of θ_p.

We wish to write an explicit expression for $\lambda^{(1)}(\pi)$ and $\lambda^{(2)}(\pi)$. We shall see that $\lambda^{(1)}(\pi)$ and $\lambda^{(2)}(\pi)$ can be expressed in terms of Dirichlet L-functions.

For exactness, we consider the case $\nu = 0$, that is,

$$\pi_{\infty}(\lambda_{\infty}) = |\lambda_{\infty}|_{\infty}^{s}.$$

We use the following notation. Let A_1 be the set of prime numbers p for which $\pi_p(x_p) = |x_p|^{s_p}$; A_2 the complementary set of prime numbers. On the basis of the formulae derived above we then have:

$$\lambda^{(1)}(\pi) = B\left(-\frac{s}{2}, \frac{1}{2}\right)\prod_{A_1}\frac{1 - p^{s_p - 1}}{1 - p^{s_p}}\prod_{A} p^{-n_p/2}\sigma,$$

where n_p is the rank of $\theta_p(x_p)$, $|\sigma| = 1$. We write this expression in another form by recalling that

$$p^{s_p} = p^{s}\theta(p),$$

where

$$\theta(p) = \prod_{q}\theta_q(p).$$

We introduce a function $\hat{\theta}(p)$ on the set of integers $n \neq 0$ by defining it as follows;

If $n = (-1)^{\varepsilon}p_1^{k_1}\cdots p_s^{k_s}$ is the decomposition of n into prime factors, then $\hat{\theta}(n) = \theta(n)$ when $p_1, \ldots, p_s$ belong to A_1, $\hat{\theta}(n) = 0$ otherwise.

It is easy to check that $\hat{\theta}(n)$ is a periodic function with the period

$$k(\hat{\theta}) = \prod_{A_2} p^{n_p},$$

that is,

$$\hat{\theta}(n+k) = \hat{\theta}(n)$$

for every integer $n \neq 0$.

Thus, the expression for $\lambda^{(1)}(\pi)$ can be written in the following form:

$$\lambda^{(1)}(\pi) = \mathrm{B}\left(-\frac{s}{2}, \frac{1}{2}\right) \prod_p \frac{1 - \hat{\theta}(p)p^{s-1}}{1 - \hat{\theta}(p)p^s} k^{-1/2}(\hat{\theta})\,\sigma,$$

where $k(\hat{\theta})$ is the period of the character $\hat{\theta}(n)$, $|\sigma| = 1$.

The product

$$L(s, \theta) = \prod_p (1 - \hat{\theta}(p)p^{-s})^{-1}$$

is called a Dirichlet L-function.

Hence, the coefficient $\lambda^{(1)}(\pi)$ can be expressed in terms of Dirichlet L-functions according to the following formula:

$$\lambda^{(1)}(\pi) = \mathrm{B}\left(-\frac{s}{2}, \frac{1}{2}\right) k^{-1/2}(\hat{\theta}) \frac{L(-s, \hat{\theta})}{L(1-s, \hat{\theta})}\,\sigma.$$

Similarly, we find

$$\lambda^{(2)}(\pi^{-1}) = \mathrm{B}\left(\frac{s}{2}, \frac{1}{2}\right) k^{-1/2}(\hat{\theta}) \frac{L(s, \bar{\hat{\theta}})}{L(1+s, \bar{\hat{\theta}})}\,\bar{\sigma}.$$

As was shown in **§ 4.12**, the square of B is the unit operator:

$$B^2 = E. \tag{12}$$

Since $B\varphi_\pi^{(1)} = \lambda^{(1)}(\pi)\varphi_{\pi^{-1}}^{(2)}$ and $B\varphi_{\pi^{-1}}^{(2)} = \lambda^{(2)}(\pi^{-1})\varphi_\pi^{(1)}$, (12) is equivalent to the following relation:

$$\lambda^{(1)}(\pi)\lambda^{(2)}(\pi^{-1}) = 1.$$

When we substitute here the explicit expressions for $\lambda^{(1)}(\pi)$ and $\lambda^{(2)}(\pi^{-1})$, we obtain

$$\mathrm{B}\left(-\frac{s}{2}, \frac{1}{2}\right)\mathrm{B}\left(\frac{s}{2}, \frac{1}{2}\right) k^{-1}(\hat{\theta}) \frac{L(-s, \hat{\theta})L(s, \bar{\hat{\theta}})}{L(1-s, \hat{\theta})L(1+s, \bar{\hat{\theta}})} = 1. \tag{13}$$

So we have established that the condition $B^2 = E$ or, what is the same, the condition of degeneracy of the operator $M = E + B$ on every space $H_\pi + H_{\pi^{-1}}$ is equivalent to the functional relation (13) for Dirichlet L-functions.

The relation (13) is a consequence of the standard functional relation for Dirichlet L-functions:

$$L(1-s, \hat{\theta}) = \tau(\hat{\theta})(2\pi)^{-s}k^{s-1/2}\Gamma(s)(e^{\pi/2si}\hat{\theta}(-1) + e^{-\pi/2si})L(s, \bar{\hat{\theta}}), \tag{14}$$

where

$$\tau(\hat{\theta}) = \begin{cases} 1, & \text{when } \hat{\theta}(-1) = 1, \\ i, & \text{when } \hat{\theta}(-1) = -1. \end{cases}$$

For in our case $\hat{\theta}(-1) = 1$. Therefore it follows from the functional relation (14) that

$$\frac{L(-s, \hat{\theta})L(s, \bar{\hat{\theta}})}{L(1-s, \hat{\theta})L(1+s, \bar{\hat{\theta}})} k^{-1}(\theta) = \left(4\Gamma(s)\Gamma(-s)\cos^2\frac{\pi}{2}s\right)^{-1}$$

$$= -\frac{s \sin \pi s}{4\pi \cos^2 \frac{\pi}{2} s} = -\frac{1}{2}\frac{s \sin \frac{\pi}{2} s}{\pi \cos \frac{\pi}{2} s}. \qquad (15)$$

On the other hand, we have

$$\mathrm{B}\left(-\frac{s}{2}, \frac{1}{2}\right)\mathrm{B}\left(\frac{s}{2}, \frac{1}{2}\right) = -\frac{2\pi \cos \frac{\pi s}{2}}{s \sin \frac{\pi s}{2}}. \qquad (16)$$

The relation (13) follows immediately from (15) and (16).

When $\nu = 1$, that is

$$\pi_\infty(\lambda_\infty) = |\lambda_\infty|^s \operatorname{sign} \lambda_\infty,$$

we have the following expressions for $\lambda^{(1)}(\pi)$ and $\lambda^{(2)}(\pi)$:

$$\lambda^{(1)}(\pi) = i\mathrm{B}\left(-\frac{s-1}{2}, \frac{1}{2}\right)\frac{L(-s, \hat{\theta})}{L(1-s, \hat{\theta})} k^{-1/2}(\hat{\theta})\sigma,$$

$$\lambda^{(2)}(\pi^{-1}) = i\mathrm{B}\left(-\frac{-s-1}{2}, \frac{1}{2}\right)\frac{L(s, \bar{\hat{\theta}})}{L(1+s, \bar{\hat{\theta}})} k^{-1/2}(\theta)\bar{\sigma}.$$

Thus, in this case the equation $B^2 = E$ turns out to be equivalent to the following relation:

$$-\mathrm{B}\left(\frac{1}{2} - \frac{s}{2}, \frac{1}{2}\right)\mathrm{B}\left(\frac{1}{2} + \frac{s}{2}, \frac{1}{2}\right) \frac{L(-s, \hat{\theta})L(s, \bar{\hat{\theta}})}{L(1-s, \hat{\theta})L(1+s, \bar{\hat{\theta}})} k^{-1}(\hat{\theta}) = 1,$$

where $\hat{\theta}(-1) = -1$.

It is easy to verify that this relation is also a consequence of the relation (14) for Dirichlet L-functions.

APPENDIX I TO § 4

Here we give the proof of the two lemmas for functions on the half-line $0 \leq \tau < \infty$. These results were used in **§ 4**.

1. Lemma on the Completeness of the Family Φ_∞. Let Φ_∞ be the family of functions $\varphi(\tau)$ defined on the half-line $0 \leq \tau < \infty$ and satisfying the following conditions:

1. The function $\varphi(\tau)$ is finite, infinitely differentiable, and vanishes in a neighborhood of 0, the same for every function φ.

2. $\displaystyle\int_0^\infty \varphi(\tau)\, d\tau = 0;$

3. $\displaystyle\int_0^\infty \varphi(\tau)\tau\, d\tau = 0.$

Let $\theta(n)$ be a fixed numerical character. We introduce the function

$$\psi(\tau) = \sum_{n=1}^{\infty} \theta(n)\,\varphi(n\tau), \qquad \varphi \in \Phi_\infty. \tag{1}$$

LEMMA. *If the function $f(\tau)$ is such that*

$$\int_0^\infty |f(\tau)|^2\, \tau\, d\tau < \infty \tag{2}$$

and $f(\tau)$ satisfies the condition

$$\int_0^\infty f(\tau)\overline{\psi(\tau)}\tau\, d\tau = 0 \tag{3}$$

for every function $\psi(\tau)$ of the form (1), *then $F(\tau) \equiv 0$.*†

Proof. First we introduce the Mellin transform of $f(\tau)$, $0 \leq \tau < \infty$, by the formula:

$$\check{f}(\rho) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(\tau)\tau^{i\rho}\, d\tau, \quad -\infty < \rho < +\infty \tag{4}$$

and prove two propositions on this Mellin transform.

PROPOSITION 1. *The Mellin transform* (4) *establishes an isometric map of the space of functions $f(\tau)$ with the norm*

$$\|f\|^2 = \int_0^\infty |f(\tau)|^2\, \tau\, d\tau.$$

† Observe that functions $\psi(\tau)$ of the form (1) are finite, infinitely differentiable, and rapidly decreasing as $\tau \to 0$.

onto the space of functions $\check{f}(\rho)$ *with the norm*

$$\|\check{f}\|^2 = \int_{-\infty}^{+\infty} |\check{f}(\rho)|^2 \, d\rho.$$

Thus,

$$\int_0^{\infty} |f(\tau)|^2 \, \tau \, d\tau = \int_{-\infty}^{+\infty} |\check{f}(\rho)|^2 \, d\rho.$$

The inversion formula for the Mellin transform has the following form:

$$f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \check{f}(\rho) \tau^{-i\rho-1} \, d\rho.$$

Proof. If we change τ to the new variable $t = \ln \tau$, then the Mellin transform goes over into the Fourier transform of a function of t, namely $f(e^t)e^t$. The assertion of the lemma follows immediately from the properties of the Fourier transform.

Proposition 2. *The Mellin transform of a function* $\psi(\tau)$ *of the form* (1) *is expressed by the following formula:*

$$\check{\psi}(\rho) = L(1 + i\rho, \theta)\check{\varphi}(\rho), \tag{5}$$

where $L(s, \theta)$ *is the Dirichlet L-function.*

Proof. We set

$$\check{\psi}(s) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \psi(\tau) \tau^{is} \, d\tau, \tag{6}$$

where s is a complex number. Since the function $\psi(\tau)$ is finite and fast decreasing as $\tau \to 0$, the integral (6) converges for any complex s and defines an entire analytic function. We substitute in (6) for $\psi(\tau)$ its expression (1). If $\operatorname{Im} s < 0$, then we may interchange summation and integration in the expression so obtained. As a result we find

$$\begin{aligned}\check{\psi}(s) &= \sum_{n=1}^{\infty} \theta(n) \cdot \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \varphi(n\tau) \tau^{is} \, d\tau \\ &= \sum_{n=1}^{\infty} \frac{\theta(n)}{n^{1+is}} \int_0^{\infty} \varphi(\tau) \tau^{is} \, d\tau.\end{aligned}$$

Thus,

$$\check{\psi}(s) = L(1 + is, \theta)\check{\varphi}(s), \quad \operatorname{Im} s < 0.$$

Since both sides of this equation can be continued analytically to the whole complex plane, the equation is valid for arbitrary s. In particular, for real s we obtain the required equation (5).

Now we proceed to the proof of the lemma. Let H be the space of all functions $f(\tau)$ satisfying conditions 2 and 3. Suppose that the lemma is not true, that is, $H \neq 0$.

Obviously, H contains functions $f \neq 0$ that are sufficiently smooth at the points $\tau \neq 0$. Let $f(\tau)$ be one of these smooth functions. Going over in (3) from the functions to their Mellin transforms we find:

$$\int_{-\infty}^{+\infty} \check{f}(\rho)\overline{L(1 + i\rho, \theta)}\overline{\check{\varphi}(\rho)}\, d\rho = 0 \tag{7}$$

for every function $\varphi \in \Phi_{\infty}$.

We introduce the new function

$$\check{F}(\rho) = \check{f}(\rho)\overline{L(1 + i\rho, \theta)} \cdot \frac{\rho}{1 + \rho^2} \tag{8}$$

We know that $L(1 + i\rho, \theta)$ has a simple pole at $\rho = 0$ and that $L(1 + i\rho, \theta) = O(\ln|\rho|)$ as $\rho \to \pm\infty$. Hence, it follows that the function $\overline{L(1 + i\rho, \theta)} \cdot \dfrac{\rho}{1 + \rho^2}$ is bounded, and therefore, $F(\rho)$ is of square-integrable modulus.

Let $F(\tau)$ be the inverse Mellin transform of $\check{F}(\rho)$. Then

$$\int_{0}^{\infty} |F(\tau)|^2\, \tau\, d\tau < \infty. \tag{9}$$

From (7) it follows that

$$\int_{0}^{\infty} F(\tau)\bar{u}(\bar{\tau})\tau\, d\tau = 0, \tag{10}$$

where $u(\tau)$ is the inverse Mellin transform of the function $\dfrac{1 + \rho^2}{\rho}\, \varphi(\rho)$, that is,

$$u(\tau) = \frac{1}{\sqrt{2\tau}} \int_{-\infty}^{+\infty} \frac{1 + \rho^2}{\rho}\, \check{\varphi}(\rho)\tau^{-i\rho-1}\, d\rho. \tag{11}$$

We have to show on the basis of (10) and (11) that $F(\tau) = 0$. For this then shows that $\check{F}(\rho) = 0$; hence, also $f(\tau) = 0$, in contradiction to the assumption made.

We express the function $u(\tau)$ defined by formula (11) directly in terms of $\varphi(\tau)$. By the formulae for the Mellin transform and the inverse Mellin transform we easily obtain:†

$$u(\tau) = -i\left(\tau^{-1}\int_0^\tau \varphi(\tau)\, d\tau - (\tau\varphi(\tau))'\right).$$

Thus, (10) assumes the following form:

$$\int_0^\infty F(\tau)\left(\int_0^\tau \overline{\varphi(\tau)}\, d\tau - \tau(\tau\overline{\varphi(\tau)})'\right) d\tau = 0 \qquad (12)$$

for every function $\varphi(\tau) \in \Phi_\infty$.

Observe now that the set Φ_∞ contains the functions $\varphi(\tau)$ of the form

$$\varphi(\tau) = \frac{d^2a(\tau)}{d\tau^2},$$

where $a(\tau)$ is an arbitrary finite function that is defined on $0 \leq \tau < \infty$ and vanishes for sufficiently small values of τ. Substituting $\varphi(\tau) = \frac{d^2a(\tau)}{d\tau^2}$ in (12) and integrating by parts we find

$$\int_0^\infty (F'(\tau) + (\tau F(\tau))'' - (\tau^2F(\tau))''')\overline{a(\tau)}\, d\tau = 0. \qquad (13)$$

Since (13) holds for every finite function $a(\tau)$ that vanishes for sufficiently small values of τ, it follows that

$$F'(\tau) + (\tau F(\tau))'' - (\tau^2F(\tau))''' = 0.$$

The general solution of this equation has the form

$$F(\tau) = C_1 + C_2 \ln \tau + C_3\tau^{-2}.$$

† It follows from the formulae for the inverse Mellin transform that

$$(\tau\varphi(\tau))' = -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \rho\check{\varphi}(\rho)\tau^{-i\rho-1}\, d\rho,$$

$$\tau^{-1}\int_0^\tau \varphi(\tau)\, d\tau = i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \rho^{-1}\check{\varphi}(\rho)\tau^{-i\rho-1}\, d\rho,$$

$$\tau^{-1}\int_0^\tau \varphi(\tau)\, d\tau - (\tau\varphi(\tau))' = iu(\tau)$$

Furthermore, since

$$\int_0^\infty |F(\tau)|^2 \tau \, d\tau < \infty,$$

it follows that $F(\tau) = 0$.

2. Lemma on Functions Defined on the Half-Line $0 \leq \tau < \infty$ **and Belonging to** L_2. Let

$$\varphi(x) = x^{-1} \sum_{\substack{n=-\infty \\ a+qn \neq 0}}^{+\infty} \psi_q(a + qn) f\left(\frac{a + qn}{x}\right), \tag{1}$$

where we use the following notation: q is a fixed positive integer; a a fixed integer from the interval $0 \leq a < q$; $\psi_q(n)$ is the multiplicative function defined by the formula

$$\psi_q(n) = \prod_{p|n} \left(1 - \frac{\chi_q(p)}{p}\right), \qquad \psi_q(\pm 1) = 1, \tag{2}$$

(the product is taken over all prime divisors of n), where

$$\chi_q(n) = \begin{cases} 1, & \text{when } (n, q) = 1, \\ 0, & \text{when } (n, q) > 1. \end{cases} \tag{3}$$

Lemma. *If $f(x)$ is a finite, infinite differentiable function on the line $-\infty < x < +\infty$ and satisfies the relation*

$$\int_{-\infty}^{+\infty} f(x) \, dx = 0, \tag{4}$$

then

$$\int_0^\infty |\varphi(x)|^2 x \, dx < \infty. \tag{5}$$

First of all we reduce the proof of the lemma to that of another proposition.

Let $(a, q) = d$ *and* $a' = \dfrac{a}{d}$, $q' = \dfrac{q}{d}$. From the definition of $\psi_q(n)$ it follows that

$$\psi_q(a + qn) = \psi_q(a' + q'n).$$

Thus, (1) may be written in the following form:

$$\varphi(x) = x^{-1} \sum_{\substack{n \in g_1, \\ n \neq 0}} \psi_q(n) f_1\left(\frac{n}{x}\right), \tag{6}$$

where $f_1(x) = f(dx)$; the summation is taken over the set g_1 of numbers of the form $a' + q'm$, $m = 0, \pm 1, \pm 2, \ldots$, that is, over a residue class mod q' to q'.

Let $G_{q'}$ be the multiplicative group of residue classes mod q' to q'. With every $g \in G_{q'}$ we associate the function

$$\varphi(x;g) = x^{-1} \sum_{\substack{n \in g, \\ n \neq 0}} \psi_q(n) f_1\left(\frac{n}{x}\right). \tag{7}$$

Our function $\varphi(x;g)$ is contained among the functions $\varphi(x)$.

Next we introduce the function

$$\varphi_\theta(x) = \sum_{g \in G_{q'}} \varphi(x;g)\theta(g), \tag{8}$$

where $\theta(g)$ is a character on $G_{q'}$. To write the expression for $\varphi_\theta(x)$ in a more convenient form we introduce the numerical character $\theta(n)$:

$$\theta(n) = \begin{cases} \theta(g), & \text{when } n \in g, \quad g \in G_{q'}, \\ 0, & \text{when } (n, q') > 1. \end{cases}$$

Then the formula for $\varphi_\theta(x)$ assumes the following form:

$$\varphi_\theta(x) = x^{-1} \sum_{n \neq 0} \theta(n) \psi_q(n) f_1\left(\frac{n}{x}\right) \tag{9}$$

or

$$\varphi_\theta(x) = x^{-1} \sum_{n=1}^{\infty} \theta(n) \psi_q(n) F\left(\frac{n}{x}\right), \tag{10}$$

where

$$F(x) = f_1(x) + \theta(-1) f_1(-x). \tag{11}$$

Observe that by the conditions imposed on $f(x)$ the function $F(x)$ is finite, infinitely differentiable on $-\infty < x < +\infty$, and that

$$\int_0^\infty F(x)\, dx = 0, \quad \text{provided that } \theta(-1) = 1. \tag{12}$$

Since the original function $\varphi(x)$ is a linear combination of functions $\varphi_\theta(x)$, where θ ranges over all characters of $G_{q'}$, the proof of the lemma reduces to that of the following assertion.

Let $F(x)$ be a finite, infinitely differentiable function defined on $0 \leq x \leq \infty$ and satisfying condition (12). Then for the function $\varphi_\theta(x)$ defined by (10) we have

$$\int_0^\infty |\varphi_\theta(x)|^2 x\, dx < \infty. \tag{13}$$

We now proceed to the proof of this assertion.

We consider the Mellin transform of $\varphi_\theta(x)$:

$$\check{\varphi}_\theta(\sigma + it) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varphi_\theta(x) x^{\sigma+it-1}\, dx. \tag{14}$$

PROPOSITION 1. *The integral* (14) *converges for* $\sigma < 0$; *the function* $\varphi(\sigma + it)$ *is analytic in the domain* $\sigma < 0$.

This follows immediately from the fact that $\varphi_\theta(x)$ vanishes for sufficiently small values of x, and that $|\varphi_\theta(x)| < C$, as $x \to \infty$.

Observe that by the inversion formula for the Mellin transform $\varphi_\theta(x)$ can be expressed in terms of $\check{\varphi}_\theta(\sigma + it)$ in the following way:

$$\varphi_\theta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \check{\varphi}_\theta(\sigma + it) x^{-\sigma - it}\, dt, \quad \sigma < 0. \tag{15}$$

PROPOSITION 2. *The function $\check{\varphi}_\theta(s)$, $s = \sigma + it$, can be expressed in terms of the Mellin transform $\check{F}(s)$ of $F(x)$:*

$$\check{\varphi}_\theta(s) = \frac{L(1 - s, \theta)}{L(2 - s, \theta)} \check{F}(1 - s), \quad \sigma < 0, \tag{16}$$

where $L(s, \theta)$ is the Dirichlet L-function:

$$L(s, \theta) = \sum_{n=1}^{\infty} \frac{\theta(n)}{n^s} = \prod_p (1 - \theta(p)p^{-s})^{-1}. \tag{17}$$

To prove this we substitute in (14) the expression for $\varphi_\theta(x)$ and find:

$$\check{\varphi}_\theta(s) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left(\sum_{n=1}^{\infty} \theta(n) \psi_q(n) F\left(\frac{n}{x}\right)\right) x^{s-2}\, dx.$$

From the fact that $F(x)$ is finite it follows that here for $\sigma < 0$ summation and integration may be interchanged. As a result we obtain

$$\check{\varphi}_\theta(s) = \left(\sum_{n=1}^{\infty} \theta(n) \psi_q(n) n^{s-1}\right) \check{F}(1 - s), \tag{18}$$

where $\check{F}(s)$ is the Mellin transform of $F(x)$.

Now we express the sum of the series in (18) in terms of the Dirichlet L-function $L(s, \theta)$:†

$$\begin{aligned}
\sum_{n=1}^{\infty} \theta(n) \psi_q(n) n^{s-1} \\
&= \prod_p (1 + \psi_q(p)\theta(p)p^{s-1} + \psi_q(p)\theta^2(p)p^{2(s-1)} + \cdots) \\
&= \prod_p \left(1 + \left(1 - \frac{\chi_q(p)}{p}\right) \frac{\theta(p)p^{s-1}}{1 - \theta(p)p^{s-1}}\right) \\
&= \prod_p \frac{1 - \theta(p)p^{s-2}}{1 - \theta(p)p^{s-1}} = \frac{L(1 - s, \theta)}{L(2 - s, \theta)}.
\end{aligned}$$

Thus formula (16) is proved.

† In the proof we use the fact that $\psi_q(1) = 1$,

$$\psi_q(p^k) = \psi_q(p) = 1 - \frac{\chi_q(p)}{p}, \qquad k = 1, 2, \ldots,$$

and that $\theta(n)\chi_q(n) = \theta(n)$ for every n.

PROPOSITION 3. *The function $\check{\varphi}_\theta(s)$, $s = \sigma + it$, defined for $\sigma < 0$ by* (16) *can be continued analytically to the whole complex plane and has no singularities in the domain $\sigma \leq 1$.*

Proof. The function $\check{F}(1 - s)$, $s = \sigma + it$, is given in the domain $\sigma < 0$ by the convergent integral

$$\check{F}(1 - s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty F(x) x^{-s}\, dx. \tag{19}$$

and, in this domain, is an analytic function of s. Integrating by parts we verify immediately that $\check{F}(1 - s)$ can be continued analytically to the whole complex s-plane and is an entire function of s.

On the other hand, it is known that the L-function $L(s, \theta)$, $s = \sigma + it$, defined for $\sigma > 1$ by the convergent integral (17), can be continued analytically to the whole complex plane.

Hence, it follows that the function $\check{\varphi}_\theta(s)$, $s = \sigma + it$, defined for $\sigma < 0$ by (16), can also be continued analytically to the whole complex s-plane.

Now we show that $\check{\varphi}_\theta(s)$ has no singularities in the domain $\sigma \leq 1$.

For this purpose we use the following well-known proposition on the poles and zeros of $L(s, \theta)$:

1. *If $\theta(g) \not\equiv 1$ $(g \in G_{q'})$, then $L(s, \theta)$ is an entire function of s. If $\theta(g) \equiv 1$, then the only singularity of $L(s, \theta)$ is a simple pole at $s = 0$.*
2. *$L(s, \theta)$, $s = \sigma + it$ has no zeros in the domain $\sigma \geq 1$.*

From these properties of $L(s, \theta)$ it follows that if $\theta(g) \not\equiv 1$, then the function

$$\check{\varphi}_\theta(s) = \frac{L(1 - s, \theta)}{L(2 - s, \theta)} \check{F}(1 - s), \quad s = \sigma + it,$$

has no singularities in $\sigma \leq 1$. But if $\theta(g) \not\equiv 1$, then its only singularity in $\sigma \leq 1$ can be a simple pole at $s = 0$ (because $L(1 - s, \theta)$ has a simple pole at $s = 0$). However, if $\theta(g) \equiv 1$, then $F(x)$ satisfies the relation

$$\int_0^\infty F(x)\, dx = 0,$$

therefore $\check{F}(1 - s)$ has a zero at $s = 0$. Consequently, $\check{\varphi}_\theta(s)$ has, in fact, no singularity at $s = 0$.

PROPOSITION 4.

$$\int_{-\infty}^{+\infty} |\check{\varphi}_\theta(\sigma + it)|^2\, dt < \text{const.}$$

in the domain $-\lambda \leq \sigma \leq 1$, where λ is any sufficiently large number.

This assertion follows immediately from the expression (16) for $\check{\phi}_\theta(\sigma + it)$ and the following estimates, as $t \to \infty$:

$$L(1 - \sigma - it, \theta) = O(|t|^a) \qquad \text{for } \sigma \le 1;$$

$$\frac{1}{L(2 - \sigma - it, \theta)} = O(|t|^a) \qquad \text{for } \sigma \le 1;$$

$$\check{F}(1 - \sigma - it) = O(|t|^{-N}).$$

Here a is a positive number; N is any sufficiently large positive number. All the estimates are uniform in σ on the interval $-\lambda \le \sigma \le 1$.

The first two of these estimates are well known in the theory of Dirichlet L-functions. The last estimate follows immediately from the properties, finiteness and infinite differentiability, of $F(x)$.

We now use the following result (see, for example, Paley and Wiener, Fourier transforms in the complex domain, Theorem IV).

Suppose that the function $\check{f}(s)$ is analytic in the strip $-\lambda \le \sigma \le \mu$, and that it satisfies in this domain the inequality

$$\int_{-\infty}^{+\infty} |\check{f}(\sigma + it)|^2 \, dt < \text{const}.$$

Then there exists a measurable function $f(y)$ such that

$$\int_{-\infty}^{+\infty} |f(y)|^2 e^{2\mu y} \, dy < \infty, \quad \int_{-\infty}^{+\infty} |f(y)|^2 e^{-2\lambda y} \, dy < \infty$$

and that on the closed interval $-\lambda < \sigma < \mu$

$$\check{f}(\sigma + it) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{(\sigma + it)y} \, dy$$

The convergence of the integral is understood in the sense of the mean square.

We apply this result to the function $\check{\phi}_\theta(s)$ (in our case $\mu = 1$). Setting $f(y) = u(e^y)$ and introducing instead of y the new variable $x = e^y$ we conclude:

There exists a measurable function $u(x)$ such that

$$\int_0^\infty |u(x)|^2 x \, dx < \infty, \quad \int_0^\infty |u(x)|^2 x^{-2\lambda - 1} \, dx < \infty \tag{20}$$

and

By the inversion formula for the Mellin transform we have

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \check{\varphi}_\theta(\sigma + it) x^{-\sigma - it}\, dt, \quad -\lambda \leq \sigma \leq 1.$$

Substituting this equation in (15) we conclude that $u(x) = \varphi_\theta(x)$. Thus, by (20) we have proved that

$$\int_0^{\infty} |\varphi_\theta(x)|^2 x\, dx < \infty.$$

and the lemma is proved.

APPENDIX II TO § 4

1. On the Connection Between the Homogeneous Space $G_Q \setminus G_A$ and the Homogeneous Spaces of the Group G_∞. Here we clarify the connection between the homogeneous space $G_Q \setminus G_A$ and the homogeneous spaces $\Gamma_m \setminus G_\infty$ of the group G_∞ of real unimodular matrices of order 2, where Γ_m is a congruence subgroup. We recall that Γ_m, where m is a natural number, consists of all integral unimodular matrices of the form

$$\gamma = e + m\gamma',$$

where e is the unit matrix and γ' an integral matrix.

Consider the space $H_m = L_2(\Gamma_m \setminus G_\infty)$. By definition it consists of the functions $f(g)$ on G_∞ that satisfy the following conditions:

1. $f(\gamma g) = f(g)$ for every $\gamma \in \Gamma_m$;
2. $\int_{\Gamma_m \setminus G_\infty} |f(g)|^2\, dg < \infty.$

Obviously, if m is divisible by n, then $\Gamma_m \subset \Gamma_n$; therefore, the inverse inclusion $H_n \subset H_m$ holds for the corresponding spaces.

Thus, the spaces H_m form a direct spectrum. We show that *the spectral limit of the spaces H_m is $L_2(X) = L_2(G_Q \setminus G_A)$.*

To prove this we establish an isomorphism between the spaces H_m and certain subspaces of $L_2(X)$ which will be defined presently.

We denote by $U^{(m)}$, where m is any natural number, the subgroup of adeles of the form

$$g = (1, u_2, \ldots, u_p, \ldots),$$

that satisfy the following conditions:

1. $u_p \in U_p$, where U_p is the subgroup of integral p-adic matrices, $p = 2, 3, \ldots$.

2. If m is divisible by p^n, then $u_p \equiv e (\mathrm{mod}\, p^n)$, where e is the unit matrix.

Clearly, the subgroups $U^{(m)}$ are compact and among them there are arbitrarily small ones, that is, every neighborhood of the unit element of G_A contains at least one such subgroup.

We denote by $L_2^{(m)}(X)$ the space of functions in $L_2(X)$ that satisfy the following condition:

$$f(gu^{(m)}) = f(g) \quad \text{for every } u^{(m)} \in U^{(m)}.$$

If m is divisible by n, then obviously $U^{(m)} \subset U^{(n)}$; hence, for the corresponding spaces $L_2^{(m)}(X)$ and $L_2^{(n)}(X)$ the inverse inclusion holds:

$$L^{(n)}(X) \subset L^{(m)}(X).$$

Thus, the spaces $L_2^{(m)}(X)$ form a direct spectrum. Their spectral limit is the whole space $L_2(X)$. This follows immediately from the fact that among the subgroups $U^{(m)}$ there are arbitrarily small ones. We show that

$$H_m \cong L_2^{(m)}(X). \tag{1}$$

As a preliminary we show that

$$\Gamma_m \setminus G_\infty \cong G_Q \setminus G_A / U^{(m)}. \tag{2}$$

To set up the isomorphism (2) we use the following result.

For every adele g and every natural number m there exists a principal adele γ for which $\gamma g = \tilde{g}_\infty u^{(m)}$, where $\tilde{g}_\infty \equiv (g_\infty, 1, 1, \ldots)$ *and* $u^{(m)} \in U^{(m)}$. This result was essentially established in **§ 4.2.** True, there we proved only the weaker proposition: *there exists a principal adele* γ *for which* $\gamma g = g_\infty u^{(1)}$. However, by modifying the arguments in **§ 4.2** slightly we can easily derive the result stated here.

From this result it follows that every double coset of $G_Q \setminus G_A / U^{(m)}$ contains representatives of the form $\tilde{g}_\infty \equiv (g_\infty, 1, 1, \ldots)$. We show that the set of elements $g_\infty \in G_\infty$, that correspond to one and the same double coset of $G_Q \setminus G_A / U^{(m)}$ forms the coset $\Gamma_m g_\infty$. Then we have the one-to-one correspondence

$$G_Q \setminus G_A / U^{(m)} \leftrightarrow \Gamma_m \setminus G_\infty. \tag{3}$$

Now two elements $\tilde{g}_\infty$ and $\tilde{g}'_\infty$ belong to one and the same double coset if and only if they are connected by a relation

$$\tilde{\gamma}\tilde{g}_\infty = \tilde{g}'_\infty u^{(m)}, \tag{4}$$

where $\tilde{\gamma} = (\gamma, \ldots, \gamma, \ldots)$ is a principal adele and $u^{(m)} \in U^{(m)}$.

The equation (4) means that

$$(1, \gamma, \ldots, \gamma, \ldots) \in U^{(m)} \tag{5}$$

$$\gamma g_\infty = g'_\infty. \tag{6}$$

But (5), as is easy to verify, is equivalent to the condition that $\gamma \in \Gamma^m$. Thus, (4) is equivalent to the condition that $\gamma g_\infty = g'_\infty$, where $\gamma \in \Gamma_m$.

So we have established a one-to-one correspondence between the points of the spaces $\Gamma_m \setminus G_\infty$ and $G_Q \setminus G_A/U^{(m)}$. An easy check shows that this is a homeomorphism.

This correspondence induces a one-to-one correspondence

$$\varphi_m: f(g_\infty) \to F(g)$$

between the functions $f(g_\infty)$, $g_\infty \in G_\infty$, that are constant on the cosets of $\Gamma_m \setminus G_\infty$, and the functions $F(g)$, $g \in G_A$, that are constant on the double cosets of $G_Q \setminus G_A/U^{(m)}$.

It is easy to verify the following properties of the map.

1. *The image of* $L_2(\Gamma_m \setminus G_\infty)$ *under* φ_m *is* $L_2^{(m)}(X)$.
2. φ_m *is an isometric map of* $L_2(\Gamma_m \setminus G_\infty)$ *onto* $L_2^{(m)}(X)$.
3. *For arbitrary natural numbers m and n, where m is divisible by n, the following diagram is commutative:*

$$\begin{array}{ccc} L_2(\Gamma_n \setminus G_\infty) & \xrightarrow{\varphi_n} & L_2^{(n)}(X) \\ \downarrow & & \downarrow \\ L_2(\Gamma_m \setminus G_\infty) & \xrightarrow{\varphi_m} & L_2^{(m)}(X). \end{array}$$

The vertical arrows indicate the embedding isomorphism.

The verification of these properties is left to the reader.

Thus, for every m we have an isometric map of $L_2(\Gamma_m \setminus G_\infty)$ onto $L_2^{(m)}(X)$. By property 3, the spectral limit of the spaces $L_2(\Gamma_n \setminus G_\infty)$ is isomorphic to that of the spaces $L_2^{(m)}(X)$, that is, to $L_2(X)$.

So we have shown that the direct spectrum of the spaces $L_2(\Gamma_m \setminus G_\infty)$ is isomorphic to $L_2(X)$.†

In fact we have a stronger result. For the space $L_2(X)$ can naturally be regarded as a module over the ring $S(G_A)$ of Schwartz-Bruhat functions on G_A. Multiplication of a function $f(g) \in L_2(x)$ by an element $\varphi(g) \in S(G_A)$ is defined by the following formula:

$$\varphi(g) * f(g) = \int f(gg')\,\varphi(g')\,dg'. \tag{7}$$

On the other hand, each of the spaces $L_2^{(m)}(X) \cong H_m$ is a module over the subring $S_m(G_A)$ of Schwartz-Bruhat functions that are constant on the double cosets of $U^{(m)} \setminus G_A/U^{(m)}$.

† We mention in passing that the spaces $\Gamma_m \setminus G_\infty \cong G_Q \setminus G_A/U^{(m)}$ also form an inverse spectrum whose spectral limit is $G_Q \setminus G_A$.

It is easy to verify that the space $L_2(X)$, regarded as a module over the ring $S(G_A)$ of Schwartz-Bruhat functions on G_A, is in a natural sense the direct spectrum of the spaces $L_2^{(m)}(X) \cong H_m$, regarded as modules over the subrings $S_m(G_A)$ of Schwartz-Bruhat functions that are constant on the double cosets of $U^{(m)} \setminus G_A/U^{(m)}$.†

In conclusion let us clarify how the elements of the ring $S_m(G_A)$ act in the space $H_m = L_2(\Gamma_m \setminus G_\infty)$.

Observe that $S_m(G_A)$ is the tensor product of two rings:

$$S_m(G_A) = S(G_\infty) \cdot S_m(G_a),$$

where $S(G_\infty)$ is the ring of functions on the group G_∞, and $S_m(G)$ the ring of functions on the group G_a of adeles of the form $(1, g_2, \ldots, g_p, \ldots)$ that are constant on the double cosets of $U^{(m)} \setminus G_a/U^{(m)}$.

Obviously the elements $\varphi \in S(G_\infty)$ act in the space

$$H_m = L_2(\Gamma_m \setminus G_\infty)$$

according to the formula

$$\varphi(g_\infty) * f(g_\infty) = \int f(g_\infty g'_\infty)\, \varphi(g'_\infty)\, dg'_\infty.$$

Therefore, we investigate how the elements of $S_m(G_A)$ act.

First, we observe that by a result stated on p. 353 every element of G_a is representable in the form

$$g = \tilde{\gamma} u^{(m)}, \tag{8}$$

where $u^{(m)} \in U^{(m)}$, $\tilde{\gamma} = (1, \gamma, \gamma, \ldots)$, and γ is a matrix over Q which is uniquely determined to within multiplication on the right by elements of Γ_m. Hence, it follows immediately that

$$U^{(m)} \setminus G_a/U^{(m)} \cong \Gamma_m \setminus G_Q/\Gamma_m$$

and therefore, that $S_m(G_A)$ is isomorphic to the ring $S(G)$ of functions on G_Q that are constant on the double cosets of $\Gamma_m \setminus G_Q/\Gamma_m$ and different from zero only on finitely many such double cosets.

---

† We give a general definition of a direct spectrum of modules.

Suppose that we are given a collection of rings R_m and a collection of R_m-modules H_m, where m ranges over a partially ordered index set. As usual, we assume that for arbitrary m_1 and m_2 there is an m with $m > m_1$, $m > m_2$. We assume that every ordered pair of indices $n < m$ is associated with a monomorphism $R_n \xrightarrow{\varphi_{nm}} R_m$ of R_n into R_m and a monomorphism $H_n \xrightarrow{\psi_{nm}} H_m$ of H_n into H_m satisfying the following conditions:

1. If $p < m < n$, then $\varphi_{pm}\varphi_{mn} = \varphi_{pn}$, $\psi_{pm}\psi_{mn} = \psi_{pn}$;
2. if $r \in R_n$, $h \in H_n$, then $\psi_{nm}(rh) = \varphi_{nm}(r)\psi_{nm}(h)$.

By virtue of 1 and 2 the monomorphisms φ_{nm} and ψ_{nm} can be interpreted as embeddings.

Let H be the spectral limit of the spaces H_m and R the spectral limit of the rings R_m. Then H can be endowed naturally with the structure of an R-module. The R-module H so obtained is called the direct spectrum of the R_m-modules H_m.

Let us write down how the ring $S_m(G_A)$ acts in the space $L_2^{(m)}(X)$. As we know, the product of a function $f \in L_2^{(m)}(X)$ by $\varphi \in S_m(G_A)$ is expressed by the following formula:

$$\varphi(g) * f(g) = \int_{G_a} f(gg')\,\varphi(g')\,dg'.$$

When we substitute here for g' the expression (8) we obtain

$$\varphi * f = \sum_{\gamma \in G_Q/\Gamma_m} f(g\tilde{\gamma}u^{(m)})\,\varphi(\tilde{\gamma}u^{(m)})\,du^{(m)} = \text{mes}\, U^{(m)} \sum_{\gamma \in G_Q/\Gamma_m} f(g\tilde{\gamma})\,\varphi(\tilde{\gamma}). \tag{9}$$

Now we go over from $L_2^{(m)}(X)$ to the space $H_m = L_2(\Gamma_m \setminus G_\infty)$ isomorphic to it. We recall that the correspondence between functions $f(g) \in L_2^{(m)}(X)$ and functions $F(g_\infty) \in L_2(\Gamma_m \setminus G_\infty)$ is realized by the formula

$$F(g_\infty) = F(\tilde{g}_\infty),$$

where $\tilde{g}_\infty = (g_\infty, 1, \ldots, 1, \ldots)$. Obviously, in this correspondence the function $f_\gamma(g) = f(g\tilde{\gamma})$ is associated with the function $F_1(g_\infty) = F(\gamma^{-1}g_\infty)$.

Thus, in the space $H_m = L_2(\Gamma_m \setminus G_\infty)$ multiplication by elements of the ring $S_m(G_Q) \cong S_m(G_a)$ is expressed by the following formula:

$$\varphi * F = c_m \sum_{\gamma \in G_Q/\Gamma_m} \varphi(\gamma) F(\gamma^{-1}g_\infty), \tag{10}$$

where

$$c = \text{mes}\, U^{(m)} = (\Gamma_1 : \Gamma_m)^{-1}.$$

Let φ_{γ_0} be the characteristic function of the double coset $\Gamma_m\gamma_0\Gamma_m$. Since every function $\varphi \in S_m(G_Q)$ is a linear combination of functions φ_{γ_0}, to prescribe a rule of multiplication by elements of $S_m(G_Q)$ it is sufficient to give it for the functions φ_{γ_0}. On the basis of the general formula (10) we have

$$\varphi_{\gamma_0} * F = c_m \sum_{\gamma \in GQ/\Gamma_m}^{(\gamma_0)} F(\gamma^{-1}g_\infty)$$

where the summation is taken over the set of cosets $\gamma\Gamma_m$ that occur in the given double coset $\Gamma_m\gamma_0\Gamma_m$. It is not hard to see that this is always a finite set.

The operator $F \to \gamma_{\gamma_0} * \Gamma$ is called a *Hecke operator*.

2. The Generalized Peterson Conjecture. In this subsection we find it convenient to consider, instead of the group of unimodular matrices of order 2, the projective group, that is, the full group of nonsingular matrices of order 2 factored by its center. We denote this group by G.

We state a conjecture about the spectrum of the space $L_2(G_Q \setminus G_A)$, which we call the generalized Peterson conjecture.

We consider an irreducible unitary representation $T(g)$ of G_A. According to **§ 3**, it is a tensor product

$$T(g) = T_\infty(g_\infty) \otimes T_2(g_2) \otimes \cdots \otimes T_p(g_p) \otimes \cdots$$

of irreducible unitary representations $T_p(g_p)$ of the groups G_p, and all the $T_p(g_p)$ except a finite number are representations of class 1.

CONJECTURE 1. *If the irreducible unitary representation*

$$T(g) = T_\infty(g_\infty) \otimes T_2(g_2) \otimes \cdots \otimes T_p(g_p) \otimes \cdots$$

belongs to the discrete part of the spectrum of $L_2(G_Q \setminus G_A)$, then among the representations $T_p(g_p)$ only finitely many can belong to the supplementary series.

We show here that for the special case when $T_\infty(g_\infty)$ is a representation of the discrete series, this conjecture is equivalent to the Peterson conjecture, which we state below. We make use of the connection between the spaces $G_Q \setminus G_A$ and $\Gamma_m \setminus G_\infty$ that was established in **§ 4** Appendix **2.1.** This connection was set up only for the group of unimodular matrices; however, all the arguments carry over without change to the group of fractional-linear transformations.

We consider automorphic forms of weight n relative to a congruence subgroup Γ_m, that is, analytic functions $f(z)$ on the half-plane $\operatorname{Im} z > 0$ that satisfy for every $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in Γ_m the condition

$$f(gz)j^{-n}(z, g) = f(z),$$

where $gz = \dfrac{\alpha z + \gamma}{\beta z + \delta}$, and $j(z, g) = \beta z + \delta$.

In Chapter 1, **§ 4** we proved that the dimension of the space of automorphic forms of weight n is finite and equal to the multiplicity with which the corresponding representation $T_n^+(g)$ of the discrete series is contained in $L_2(\Gamma_m \setminus G_\infty)$.

With every double coset $\Gamma_m \gamma \Gamma_m$ of Γ_m in the group of matrices with elements from the field of rational numbers we associate an operator $S_\gamma^{m,n}$ in the space of automorphic forms of weight n relative to Γ_m:

$$S_\gamma^{m,n} f(z) = \frac{1}{n_\gamma} \sum_{\gamma_i \in \Gamma_m \setminus G_a} f(\gamma_i z)\, j^{-n}(z, \gamma_i), \tag{1}$$

where the sum is taken over the set of cosets $\Gamma_m \gamma_i$ that occur in the given double coset $\Gamma_m \gamma \Gamma_m$; n_γ is the number of the cosets. As we mentioned in **§ 4** Appendix **2.1**, this set is always finite. The operators $S_\gamma^{m,n}$ are called *Hecke operators* in the space of automorphic

forms. It is not hard to verify that Hecke operators carry automorphic forms again into automorphic forms.

In what follows we consider only Hecke operators corresponding to the matrices

$$\gamma_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

where p is a prime number not dividing m. For brevity, we denote these operators by S_p.

It is not hard to verify that the operators S_p are self-adjoint and that they commute with each other. Thus, the space of automorphic forms can be decomposed into a direct sum of one-dimensional spaces that are invariant under the S_p. Let $\lambda_p^{(1)}, \ldots \lambda_p^{(s)}$ (s is the dimension of the space of automorphic forms) be the eigenvalues of the Hecke operators S_p on these subspaces.

Conjecture 2 (Peterson). *For all prime numbers p, with the possible exception of finitely many, the following estimate holds for the eigenvalues of the Hecke operators S_p:*

$$|\lambda_1^{(k)}| < 2\sqrt{p}, \qquad k = 1, \ldots, s.$$

Here we establish a connection between the Conjectures 1 and 2: Conjecture 1, for the special case when $T_\infty(g_\infty)$ is the representation of the discrete series with the index n, is equivalent to the Peterson conjecture for the space of automorphic forms of weight n.

For this purpose we set up a correspondence between the irreducible representations $T(g)$ of G_A belonging to the discrete spectrum of $L_2(G_Q \setminus G_A)$ and automorphic forms.

Suppose then that

$$T(g) = T_\infty(g_\infty) \otimes T_2(g_2) \otimes \cdots \otimes T_p(g_p) \otimes \cdots$$

is an irreducible representation of G_A belonging to the discrete spectrum of $L_2(G_Q \setminus G_A)$ and that $H \subset L_2(G_Q \setminus G_A)$ is the subspace in which this representation acts. We also assume that $T_\infty(g_\infty)$ is the representation of the discrete series with the index n.

First, we deal with the case when all the representations $T_2(g_2), \ldots, T_p(g_p), \ldots$ are of class 1, that is, each of them is a representation of the fundamental or supplementary series.

We consider the projection operator

$$p_1 = \int_{U^{(1)}} T(u)\, du,$$

where the integral is taken over the subgroup $U^{(1)}$ of adeles of the form $(1, u_2, \ldots, u_p, \ldots)$, $u_p \in U_p$, the subgroup of integral p-adic matrices. The operator p_1 projects $H \subset L_2(G_Q \setminus G_A)$ into a subspace

$H^{(1)} \subset L_2^{(1)}(G_Q \setminus G_A)$ (for the notation see **§ 4** Appendix **2.1**). Under the isomorphism

$$L_2^{(1)}(G_Q \setminus G_A) \cong L_2(\Gamma_1 \setminus G_\infty)$$

$H^{(1)}$ corresponds to a subspace $\tilde{H}^{(1)} \subset L_2(\Gamma_1 \setminus G_\infty)$.

It is not difficult to verify that $\tilde{H}^{(1)}$ is an invariant irreducible subspace of $L_2(\Gamma_1 \setminus G_\infty)$, regarded as a module over the ring $S(G_\infty) \otimes S_1(G_Q)$, where $S_1(G_Q)$ is the ring of functions on G_Q that are constant on the double cosets of $\Gamma_1 \setminus G_Q / \Gamma_1$. Here the representation of G_∞ acting in $\tilde{H}_1^{(1)}$ belongs to the discrete series and has the index n.

By the duality theorem, $\tilde{H}^{(1)}$ is associated with a one-dimensional subspace of automorphic forms. This subspace of automorphic forms has the following properties, the verification of which is left to the reader:

1. It is invariant under the Hecke operators S_p;
2. The eigenvalues of the operators S_p on this subspace are expressed by the following formula:

$$\lambda_p = \nu_p \varphi(\gamma_p), \tag{2}$$

where ν_p is the number of cosets $\Gamma_1 \gamma$ contained in the given double coset $\Gamma_1 \gamma_p \Gamma_1$, $\gamma_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, and $\varphi_p(g_p)$ is the elementary spherical function corresponding to the representation $T_p(g_p)$.

On the basis of (2) it is not hard to obtain an explicit expression for λ_p. First of all we note that ν_p is equal to the index

$$\nu_p = \Gamma_1 : \Gamma^{(p)}$$

of the subgroup $\Gamma^{(p)} = \gamma_p \Gamma_1 \gamma_p^{-1} \cap \Gamma_1$ in Γ_1.† The subgroup $\Gamma^{(p)}$ consists of all the integral matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in which c is a multiple of p. Consequently, (Chapter 1, Appendix, p. 114),

$$\nu_p = 1 + p.$$

On the other hand, by a simple calculation, similar to that in Chapter 2, **§ 3.10** for the unimodular group, we find

$$\varphi(\gamma_p) = \frac{\sqrt{p}}{p+1}\,(p^{s_p} + p^{-s_p}) = \frac{2\sqrt{p}}{p+1} \cosh\,(s_p \ln p),$$

where s is the "index" of $T_p(g_p)$, that is, an imaginary number for a representation of the fundamental series and a real number in the

† There is a one-to-one correspondence $\gamma\Gamma^{(p)} \leftrightarrow \gamma\gamma_p\Gamma_1$ between the cosets of $\Gamma_1/\Gamma^{(p)}$ and the cosets of G_Q/Γ_1 that belong to the double coset $\Gamma_1\gamma_p\Gamma_1$.

interval $-1 < s < 1$ for a representation of the supplementary series.

Hence, the eigenvalue of a Hecke operator on a one-dimensional subspace of automorphic forms is expressed by the following formula:

$$\lambda_p = 2\sqrt{p} \cosh (s_p \ln p). \tag{3}$$

In this way we associate with every irreducible representation

$$T(g) = T_\infty(g_\infty) \otimes T_2(g_2) \otimes \cdots \otimes T_p(g_p) \otimes \cdots$$

of G_A that belongs to the discrete part of the spectrum of $L_2(G_Q \setminus g_A)$, where $T_\infty(g_\infty)$ is the representation of the discrete series with the index n and $T_2(g_2), \ldots, T_p(g_p), \ldots$ are representations of class 1, a one-dimensional space of automorphic forms of weight n relative to the modular group Γ_1. This space is invariant under the Hecke operators S_p, and the eigenvalues λ_p of the operators S_p are expressed by (3) in terms of the index s_p of the representation $T_p(g_p)$.

By carrying out our construction in the opposite order we can construct, for a given one-dimensional subspace of automorphic forms that is invariant under the Hecke operators, an irreducible representation $T(g) = T_\infty(g_\infty) \otimes T_2(g_2) \otimes \cdots \otimes T_p(g_p) \otimes \cdots$ of G_A belonging to the discrete spectrum of $L_2(G_Q \setminus G_A)$, where $T_\infty(g_\infty)$ is the representation of the discrete series with the index n, and $T_2(g_2), \ldots, T_p(g_p)$ are representations of class 1.

A similar construction holds for a representation

$$T(g) = T_\infty(g_\infty) \otimes T_2(g_2) \otimes \cdots \otimes T_p(g_p) \otimes \cdots,$$

where some of the $T_p(g_p)$ are not of class 1. In this case we can always find an m such that in the representation space of $T(g)$ there exists a vector that is invariant under the operators $T(u)$, $u \in U^{(m)}$. (The definition of the subgroups $U^{(m)}$ is on p. 352.)

By repeating the preceding construction, we can associate with this representation an automorphic form relative to Γ_m that is an eigenfunction of the Hecke operators S_p, where p ranges over the set of prime numbers not dividing m. Here the eigenvalues of the operators S_p corresponding to this form are expressed, as before, by (3). Conversely, if such an automorphic form is given, then we can use it to construct a unique irreducible representation of G_A that belongs to the discrete spectrum of $L_2(G_Q \setminus G_A)$ and contains a vector invariant under $U^{(m)}$.

From the correspondence we have established between the representation $T(g)$ and the automorphic forms, and from the formulae (3) for the eigenvalues of the Hecke operators, it follows immediately that the Peterson conjecture and the special case of conjecture 1 are equivalent. To see this, we need only observe that

by (3) the inequality

$$|\lambda_p| < 2\sqrt{p} \tag{4}$$

holds if and only if the "index" s of the representation $T_p(g_p)$ is an imaginary number, that is, that the representation belongs to the fundamental series.

Thus, the Peterson conjecture that (4) holds for all primes p, except finitely many, is equivalent to the assertion that in the corresponding representation

$$T(g) = T_\infty(g_\infty) \otimes T_2(g_2) \otimes \cdots \otimes T_p(g_p) \otimes \cdots$$

of G_A all representations $T_p(g_p)$, except finitely many, belong to the fundamental series.

§ 5. THE SPACE OF HOROSPHERES

1. Reductive Algebraic Groups. Let G be a linear algebraic group defined over the field of rational numbers Q. The group G is called *reductive* if it contains no nontrivial unipotent normal subgroups that are connected as algebraic varieties.†

If G contains no nontrivial connected solvable normal subgroups, then it is called semisimple. Every reductive group is a direct product of a semisimple group and a certain torus, that is, a commutative group of matrices reducible to diagonal form over the field of complex numbers.

We quote without proofs some fundamental properties of reductive groups. For a detailed treatment of these questions we refer the reader to the papers [3, 5, 7].

We denote by Z a maximal connected unipotent subgroup of G defined over Q. Note that all the maximal unipotent subgroups are conjugate.

Let G' denote the normalizer of Z, that is, the set of elements g' for which $Zg' = g'Z$. Obviously, G' is an algebraic subgroup of G also defined over Q.

Clearly, Z is a maximal connected unipotent normal subgroup of G'. Hence, it follows that G' can be represented as a semidirect product

$$G' = DZ. \tag{1}$$

where D is a reductive group. Here all the elements of D are semisimple, that is, reducible to diagonal form.

We denote by N the normalizer of D. It can be shown that the factor group

$$S = N_Q/D_Q \tag{2}$$

† A group of matrices is called *unipotent* if all the eigenvalues of the matrices are 1.

is always finite. This is called the *Weyl group*. Every element s of the Weyl group determines an automorphism of D:

$$\delta \to s^{-1}\,\delta s.$$

It can be shown that the following decomposition holds:

$$G_Q = Z_Q N_Q Z_Q. \tag{3}$$

In some papers (3) is called the generalized *Bruhat Lemma*.

This decomposition for the classical complex groups was obtained by Gel'fand and Naimark who first established the fundamental role of this decomposition in the theory of representations. The decomposition became more widely known after Harish-Chandra also proved (3) for real semisimple groups. More general investigations on this problem were made by Tits and Borel.

Now let T be a maximal torus splitting over Q in D. It is easy to verify that T lies in the center of D and that the Weyl group S carries the torus into itself.

Let $\mathfrak{G}$ be the Lie algebra of g, and $\mathfrak{T}$ the Lie algebra of T. With every $t \in \mathfrak{T}$ we associate the linear transformation ad t in the space $\mathfrak{G}$: ad $t: \mathfrak{g} \to [t, \mathfrak{g}]$, the adjoint representation of $\mathfrak{T}$.

The algebra $\mathfrak{G}$ may be represented as a direct sum

$$\mathfrak{G} = \sum \mathfrak{G}_\alpha \tag{4}$$

of subspaces $\mathfrak{G}_\alpha$ on which the operators ad t are multiples of the unit operator, that is,

$$[t, \mathfrak{g}_\alpha] = \alpha(t)\mathfrak{g}_\alpha$$

for every $t \in \mathfrak{T}$ and every $\mathfrak{g}_\alpha \in \mathfrak{G}_\alpha$. Here the $\alpha(t)$ are linear functions on $\mathfrak{T}$.

The subspace $\mathfrak{G}_0$ corresponding to $\alpha(t) \equiv 0$ is the Lie algebra of D. This subspace can be represented as a sum

$$\mathfrak{G}_0 = \mathfrak{T} + \mathfrak{C},$$

where $\mathfrak{C}$ is the complement of $\mathfrak{T}$ in $\mathfrak{G}_0$. Thus, the decomposition (4) can be written in the following form:

$$\mathfrak{G} = \mathfrak{T} + \mathfrak{C} + \sum_{\alpha \neq 0} \mathfrak{G}_\alpha. \tag{5}$$

We note that the operators ad $\mathfrak{g}_0$, $\mathfrak{g}_0 \in \mathfrak{G}_0$, also carry each space $\mathfrak{G}_\alpha$ into itself.

The nontrivial linear functions $\alpha(t)$ arising in (4) are usually called *roots*. We denote the set of all roots by Σ.

Let E denote the space of all linear functions defined over Q. Clearly, the Weyl group S acts in a natural way in E.

If G is a semisimple group, then the following propositions hold:

1. *In E there is a scalar product (ξ, η) that is invariant under the Weyl group S.*

2. *Among the vectors Σ there are precisely n linearly independent ones, where n is the dimension of E.*

3. *For any two roots, $\alpha, \beta \in \Sigma$ the quotient*

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

is an integer.

4. *The system Σ is invariant under the maps corresponding to the roots $\alpha \in \Sigma$, that is, the transformations*

$$\beta \to \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \alpha.$$

We introduce a lexicographic ordering in E. First, we take an arbitrary system of coordinates in E and say that $\alpha > \beta$ if the first nonzero coordinate of the vector $\alpha - \beta$ is positive.

We call a root α positive if $\alpha > 0$ and negative if $\alpha < 0$. Thus, the set of all roots splits into positive and negative roots. We call a positive root simple if it cannot be represented as the sum of two positive roots. Then the following propositions hold:

5. *The simple roots are linearly independent and their number is the dimension of E.*

6. *Every positive root is the sum of simple roots.*

7. *The maps corresponding to simple roots generate the whole Weyl group.*

8. *The subalgebra $\mathfrak{Z} = \sum_{\alpha > 0} \mathfrak{G}_\alpha$, where the sum is taken over all positive roots, is a maximal nilpotent subalgebra of G.*

We call a reductive group G *splitting* if the dimensions of maximal tori in G that split over Q and over C are the same.

2. The Space $L_2(D_Q Z_A \setminus G_A)$. Let G be an algebraic reductive group defined over the field Q of rational numbers. We denote by Z a maximal unipotent subgroup of it and by G' the normalizer of Z in G. As we mentioned in **§ 5.1**, this normalizer decomposes into the semidirect product $G' = DZ$ of the normal subgroup Z and a reductive subgroup D, whose elements are all semisimple.

In this subsection we arrive at the decomposition into irreducible representations of a representation of G_A generated by

$$\Omega = D_Q Z_A \setminus G_A,$$

that is, in $L_2(\Omega)$. The case when G_A is the group of matrices of order 2 over A was discussed in **§ 4.**

A similar problem was solved in Chapter 1. There we found the decomposition of the representation of the group G_∞ of real matrices generated by the space $Z_\infty \setminus G_\infty$. The subgroup of diagonal matrices played a fundamental role. For we established a one-to-one correspondence between the characters, that is, the one-dimensional representations, of the subgroup of diagonal matrices and the irreducible representations of G_∞ occurring in $L_2(Z_\infty \setminus G_\infty)$.

We shall see that in our case the situation is similar. The fundamental role in the decomposition of a representation G_A in $L_2(\Omega)$ is played by the group D_A, which is analogous to the group of diagonal matrices.

To begin with we consider the problem of decomposing the representation of D_A generated by $D_Q \setminus D_A$ into irreducible representations. This representation acts in the space $H = L_2(D_Q \setminus D_A)$ of functions $f(\delta)$, $\delta \in D_A$, satisfying the following conditions:

1. $f(\delta_Q\delta) = f(\delta)$ for every $\delta_Q \in D_Q$;
2. $\displaystyle\int_{D_Q\setminus D_A} |f(\delta)|^2\, d\delta < \infty.$

The representation operators $T(\delta)$ are translation operators:

$$T(\delta_0)f(\delta) = f(\delta\delta_0).$$

Let K be the center of D. We consider the characters $\pi(k)$ on K_A that are identically equal to unity on K_Q. With every character $\pi(k)$ we associate the space H_π of functions $f_\pi(\delta)$ satisfying the following conditions:

1. $f_\pi(\delta_Q\delta) = f_\pi(\delta)$ for every $\delta_Q \in D_Q$;
2. $f_\pi(k\delta) = \pi(k)f_\pi(\delta)$ for every $k \in K_A$;
3. $\displaystyle\|f_\pi\|_\pi^2 = \int_{D_Q\setminus D_Q/K_A} |f_\pi(\delta)|^2\, d\delta < \infty.$

It is not hard to see that $H = L_2(D_Q \setminus D_A)$ *splits into the continuous direct sum of the spaces* H_π*:*

$$H = \int H_\pi\, d\pi. \tag{1}$$

This decomposition is realized by the following formulae:

$$f(\delta) = \int f_\pi(\delta)\, d\pi; \tag{2}$$

$$\|f\|^2 = \int \|f_\pi\|_\pi^2\, d\pi, \tag{3}$$

where the integration is taken with respect to the invariant measure $d\pi$ on the group of characters π. Here the component f_π of the

function $f \in H$ in H_π is given by the following formula:

$$f_\pi(\delta) = \int\limits_{K_Q \setminus K_A} \bar{\pi}(k) f(k\delta)\, dk. \tag{4}$$

It remains to decompose each of the spaces H_π into irreducible subspaces.

From the fact that all the elements of D are semisimple it follows that the space $D_Q \setminus D_A / K_A$ is compact (see **§ 6.1**). It follows, as we shall show later in **§ 6.2**, that H splits into the direct sum of a countable number of invariant irreducible subspaces $H_\pi^{(n)}$:

$$H_\pi = \sum_n H_\pi^{(n)}. \tag{5}$$

In what follows we assume that the decomposition of $H = L_2(D_Q \setminus D_A)$ into the direct sum of the spaces $H_\pi^{(n)}$ (formulae (1) and (5)) is known to us.

We show that this decomposition induces a decomposition of the space $\tilde{H} = L_2(D_Q Z_A \setminus G_A)$ into a direct sum of invariant subspaces

$$\tilde{H}_\pi^{(n)} : \tilde{H} = \int \tilde{H}_\pi \, d\pi; \quad \tilde{H}_\pi = \sum_n \tilde{H}_\pi^{(n)}.$$

As a preliminary, we introduce a convenient realization of $\tilde{H}$. We examine the homogeneous space $Y = Z_A \setminus G_A$. Note that if we multiply each coset $y = Z_A g$ on the left by an element $\delta \in D_A$, it goes over into another coset, which we denote by δg. Thus, the elements $\delta \in D_A$ give transformations in Y

$$y \to \delta y,$$

which we call *left translations*.

Clearly, left translations commute with the transformations of G_A, that is,

$$\delta(yg) = (\delta y) g.$$

In terms of left translations $\tilde{H} = L_2(D_Q Z_A \setminus G_A)$ can be defined as the space of the functions $f(y)$ on $Y = Z_A \setminus G_A$ that satisfy the following conditions:

1. $f(\delta_Q y) = f(y)$ for every $\delta_Q \in D_Q$;
2. $\|f\|^2 = \int\limits_{D_Q \setminus Y} |f(y)|^2 \, dy < \infty.$

Now we introduce a measure in $D_A \setminus Y$. Let dy be an invariant measure on Y relative to the transformations of G_A. Then $d_1 y = d(\delta y)$ is also an invariant measure on Y for every $\delta \in D_A$, hence, proportional to dy. We denote the factor of proportionality

by $\beta(\delta)$. Thus, we have by definition

$$d(\delta y) = \beta(\delta)\, dy. \tag{6}$$

From the definition it follows that the function $\beta(\delta)$ is a character on D_A, that is

$$\beta(\delta_1\delta_2) = \beta(\delta_1)\beta(\delta_2) \text{ for any } \delta_1, \delta_2 \in D_A.$$

Note that $\beta(\delta_Q) = 1$ for every $\delta_Q \in D_Q$.

On y we give a nonnegative function $\rho(y)$ satisfying the following conditions:

1. The functions $\rho(y)$ and $\rho^{-1}(y)$ are measurable and summable on every compact subset.

2. $\rho(\delta y) = \beta(\delta)\rho(y)$ for every $\delta \in D_A$.

It is not hard to verify that such functions $\rho(y)$ always exist.

We give a measure $d\tilde{y}$ in $D_A \setminus Y$ by means of the following integral relation:

$$\int\limits_{D_Q\setminus D_A} f(y)\rho^{-1}(y)\, dy = \int\limits_{D_A\setminus Y}\int\limits_{D_Q\setminus D_A} f(\delta y)\, d\delta\, d\tilde{y}, \tag{7}$$

where $f(y)$ is any function summable on Y and satisfying the condition $f(\delta_Q y) = f(y)$ for every $\delta_Q \in D_Q$.

From the relation (7) it follows immediately that under the group translation $\tilde{y} \to \tilde{y}g$ in $D_A \setminus Y$ the measure $d\tilde{y}$ transforms according to the following formula:

$$d(\tilde{y}g) = \left[\frac{\rho(yg)}{\rho(y)}\right]^{-1} d\tilde{y} \tag{8}$$

Thus, $d\tilde{y}$ is not, in general, an invariant, but only a quasi-invariant measure (an invariant measure in Y need not even exist).

From the integral relation (7) we derive the following expression for the norm $\|f(y)\|$ of f in $\tilde{H}$:

$$\|f\|^2 = \int\limits_{D_A\setminus Y}\int\limits_{D_Q\setminus D_A} |f(\delta y)|^2\, \beta(\delta)\rho(y)\, d\delta\, d\tilde{y}. \tag{9}$$

It follows that

$$\int\limits_{D_A} |f(\delta y)|^2\, \beta(\delta)\, d\delta < \infty$$

for almost all $y \in Y$. Moreover, since

$$f(\delta_Q\, \delta y) = f(\delta y) \text{ for every } \delta_Q \in D_Q,$$

we have shown that $\beta^{1/2}(\delta)f(\delta y)$, regarded as a function of δ, belongs to the space $H = L_2(D_Q \setminus D_A)$.

Once we are given the decomposition of H into irreducible subspaces $H_\pi^{(N)}$, it is easy to derive the decomposition of

$$\tilde{H} = L_2(D_Q Z_A \setminus G_A).$$

We denote by $\tilde{H}_\pi^{(n)}$ the space of functions $f(y), y \in Y$, that satisfy the following conditions:

1. $f(\delta_q y) = f(y)$ for every $\delta_q \in D_Q$.

2. $\beta^{1/2}(\delta) f(\delta y)$, regarded as a function of $\delta \in D_A$, belongs to $H_\pi^{(n)}$ for almost all y.

3. $\int\limits_{D_A \backslash Y} \|\beta^{1/2}(\delta) f(\delta y)\|^2 \, \pi(y) \, d\tilde{y} < \infty$, where $\| \quad \|_\pi$ denotes the norm in H_π.

A representation of G_A acts in a natural manner in $H_\pi^{(n)}$. We say that it is *induced* by the representation of D_A in $H_\pi^{(n)}$.

From (9) it follows immediately that $H = L_2(D_Q Z_A \setminus G_A)$ splits into the spaces $\tilde{H}_\pi(n)$:

$$\tilde{H} = \int \tilde{H}_\pi, \qquad \text{where } \tilde{H}_\pi = \sum_n \tilde{H}_\pi^{(n)}.$$

Here the component $f_\pi^{(n)}(y)$ of the vector $f(y) \in \tilde{H}$ is defined as follows. Let $f_\pi^{(n)}(\delta, y)$ be the component in $H_\pi^{(n)}$ of the function $\beta^{1/2}(\delta) f(\delta y)$, regarded as a function of δ for fixed y. Then

$$f_\pi^{(n)}(y) = f_\pi^{(n)}(1, y).$$

Now we must ascertain which representations $\tilde{H}_\pi^{(n)}$ are equivalent.

First of all, we observe that the representations in $\tilde{H}_{\pi_1}^{(m)}$ and $\tilde{H}_{\pi_2}^{(n)}$ are inequivalent when $\pi_1 \neq \pi_2$. This follows from the fact that the representation operators corresponding to elements $k \in K_A$ are given in $\tilde{H}_\pi^{(n)}$ by the following formula:

$$T(k) f = \pi(k) f.$$

Thus, it is obvious that for distinct π even the representations of the subgroup K_A in the spaces $\tilde{H}_\pi^{(n)}$ are inequivalent.

Hence, it suffices to find the condition of equivalence of representations in the spaces $\tilde{H}_\pi^{(n_1)}$ and $\tilde{H}_\pi^{(n_2)}$. We begin by introducing the concept of a representation in general position.

We consider the Weyl group S of G. Every element $s \in S$ is an automorphism

$$\delta \to \delta^s$$

of D_A. Clearly, if $\tau(\delta)$ is some representation of D_A, then

$$\tau^s(\delta) \equiv \tau(\delta^s)$$

is also a representation of D_A.

We say that an irreducible representation $\tau(\delta)$ of D_A is *in general position* if the representations $\tau^s(\delta)$, $s \in S$, are pairwise inequivalent. We also say that a representation $T_\tau(g)$ of G_A is in

general position if it is induced by an irreducible representation $\tau(\delta)$ of D_A in general position.

For reductive groups that split over Q the following two propositions hold:

1. *Representations of G_A in general position are irreducible.*

2. *Two representations in general position in the spaces $\tilde{H}_\pi^{(n_1)}$ and $\tilde{H}_\pi^{(n_2)}$ induced, respectively, by representations $\tau_1(\delta)$ and $\tau_2(\delta)$ of D_A, are equivalent if and only if*

$$\tau_1 = \tau_2^s$$

for some element $s \in S$.

We do not prove these propositions here. The proof is close to standard arguments in representation theory; see Gel'fand and Naimark [29], and especially Bruhat [10, 11, 12].

From Proposition 2 it follows immediately that each representation $T_\tau(g)$ in general position occurs in $L_2(D_Q Z_A \setminus G_A)$ with multiplicity equal to the order of the Weyl group S.

3. The Operators B_s. Again, let G be a reductive group defined over Q, Z a maximal unipotent subgroup of it, D a reductive subgroup of G such that DZ is the normalizer of Z, N the normalizer of D.

We introduce two important subgroups $\tilde{Z}^n$ and Z^n of Z, corresponding to every fixed element $n \in N_Q$.

Let $\mathfrak{Z}$ be the Lie algebra of Z, that is,

$$\mathfrak{Z} = \sum_{\alpha>0} \mathfrak{G}_\alpha, \tag{1}$$

where the summation is taken over the set of all positive roots. For every $n \in N_Q$ we set

$$\tilde{\mathfrak{Z}}^n = \sum_{\alpha>0,\alpha^n>0} \mathfrak{G}_\alpha, \tag{2}$$

$$\mathfrak{Z}^n = \sum_{\alpha>0,\alpha^n<0} \mathfrak{G}_\alpha, \tag{3}$$

where α^n denotes the result of applying the element n of N_Q to the root α, that is, $\alpha^n(t) = \alpha(t^n)$, $t \in \mathfrak{T}$. The first sum is taken over the set of roots $\alpha > 0$ for which $\alpha^n > 0$ and the second over the set of roots $\alpha < 0$ for which $\alpha^n < 0$. In particular, if $n \in D$, then $\tilde{\mathfrak{Z}}^n = \mathfrak{Z}$, $\mathfrak{Z}^n = 0$. Obviously, $\tilde{\mathfrak{Z}}^n$ and $\mathfrak{Z}^n$ are disjoint subalgebras of $\mathfrak{Z}$, and

$$\tilde{\mathfrak{Z}} = \mathfrak{Z}^n + \mathfrak{Z}^n. \tag{4}$$

We denote by $\tilde{Z}^n$ and Z^n the subgroups of Z corresponding, respectively, to $\tilde{\mathfrak{Z}}^n$ and $\mathfrak{Z}^n$. In particular, for $n \in D$ we have $\tilde{Z}^n = Z$, $Z^n = 1$.

From (4) it follows that

$$Z = \tilde{Z}^n Z^n.$$

For every element z of Z has a unique representation as a product

$$z = \tilde{z}^n z^n,$$

where $\tilde{z}^n \in \tilde{Z}^n$, $z^n \in Z^n$.

It is not hard to verify that $\tilde{Z}^n$ can be defined directly, without recourse to the Lie algebra, by the following formula:

$$\tilde{Z}^n = Z \cap nZn^{-1}.$$

We note that the set of the subgroups $\tilde{Z}^n$ is the same as that of the Z^n. For the Weyl group is known to contain an element s_0 carrying all the positive roots into negative ones. Clearly, the condition $\alpha^n < 0$ is equivalent to $\alpha^{ns_0} > 0$. Consequently,

$$Z^n = \tilde{Z}^{ns_0}.$$

We denote by G_A, Z_A and so forth, the group of adeles of G, Z, and so forth, and by G_Q, Z_Q and so forth the subgroups of principal adeles.

With every $n \in N_Q$ we associate the operator B_n in the space of functions on $\Omega = D_Q Z_A \setminus G_A$ that is defined by the following formula:

$$B_n f(y) = \int_{Z_A^n} f(y_0 n^{-1} z g)\, dz. \tag{5}$$

Here y_0 denotes the point of Ω corresponding to the unit coset of $D_Q Z_A$; the integration is taken with respect to the invariant measure on Z_A^n.

It is not hard to verify that the integral necessarily converges if $f(y)$ is a finite function on Ω.

We call the B_n *Weyl operators* in the space of functions Ω.

From the definition it follows immediately that

$$B_{n0} = B_n \quad \text{for every } \delta \in D_Q$$

Thus, the operators B_n give, in fact, the elements of the Weyl group $S = N_Q/D_Q$. Therefore, in what follows we often write B_s instead of B_n, where s stands for an element of the Weyl group.

We show that the function

$$f_1(g) = B_n f(y)$$

is constant on the coset $D_Q Z_A G_A$, *that is,*

$$f_1(\delta z g) = f_1(g) \quad \text{for any } \delta \in D_Q,\ z \in Z_A. \tag{6}$$

Thus, it can be regarded as a function on Ω.

Proof. As a preliminary we show that

$$dz_A^n = d(\delta^{-1}z_A^n\delta) \quad \text{for every } \delta \in D_Q. \tag{7}$$

For when we denote by $\chi(\delta)$ the determinant of the transformation

$$z \to \delta^{-1}z\delta, \quad \text{where } \delta \in D_Q,\ z \in Z_Q,$$

then $\chi(\delta)$ is clearly a character of D_Q with values in Q^*. As we know from **§ 1**,

$$dz_A^n = dz_\infty^n\, dz_2^n \cdots dz_p^n \cdots$$

where dz_p^n is the invariant measure on Z_p^n. Since under the transformation $z_p^n \to \delta^{-1}z_p^n\delta$ the measure dz_p^n is multiplied by $|\chi(\delta)|_p$, under the map $z_A^n \to \delta^{-1}z_A^n\delta$ the measure dz_A^n is multiplied by $\prod_p |\chi(\delta)|_p = |\chi(\delta)| = 1$ because $\chi(\delta)$ is a principal idele.

From (7) it follows at once that

$$\begin{aligned} f_1(\delta g) &= \int_{Z_A^n} f(y_0 n^{-1}z\,\delta g)\, dz \\ &= \int_{Z_A^n} f(y_0(n^{-1}\,\delta n)n^{-1}zg)\, dz \\ &= \int_{Z_A^n} f(y_0 n^{-1}zg)\, dz. \end{aligned}$$

Here we use the fact that $n^{-1}\,\delta n \in D_Q$; hence, $y_0(n^{-1}\,\delta n) = y_0$.

So we have shown that

$$f_1(\delta g) = f_1(g) \quad \text{for every } \delta \in D_Q.$$

Now we show that

$$f_1(z_0 g) = f_1(g)$$

for every $z_0 \in Z_A$. We split the element z_0 into the product

$$z_0 = \tilde{Z}_0^n z_0^n$$

where $\tilde{z}_0^n \in \tilde{Z}_A^n$, $z_0^n \in Z_A^n$. Then we have

$$f_1(z_0 g) = \int_{Z_A^n} f(y_0(n^{-1}\tilde{z}_0^n n)n^{-1}z_0^n zg)\, dz.$$

Note that $n^{-1}Z_0^n n \in Z_A$; hence, $y_0(n^{-1}\tilde{z}_0 n) = y_0$. Consequently,

$$f_1(z_0 g) = \int_{Z_A^n} f(y_0 n^{-1}z_0^n zg)\, dz = \int_{Z_A^n} f(y_0 n^{-1}zg)\, dz = f_1(g).$$

Here we have used the invariance of the measure dz. The relation

(6) is now completely proved, and we have shown that

$$f_1(g) = B_n f(y)$$

is a function on $\Omega = D_Q Z_A \setminus G_A$.

So we have established that the Weyl operators B_n defined by the formula

$$B_n f(y) = \int\limits_{Z_A^n} f(y_0 n^{-1} z g)\, dz, \tag{8}$$

carry functions on $\Omega = D_Q Z_A \setminus G_A$ into functions on Ω.

We emphasize that so far the formula defines B_n only on differentiable functions in Ω that are finite or decrease sufficiently fast.

4. Properties of the Operators B_s. First we mention two important properties of the operators B_s that follow immediately from the definition:

1. The operators B_s commute with the representation operators.
2. The operator B_s carries every irreducible representation $T_\tau(g)$ induced by a representation $\tau(\delta)$ of D_A into a representation $T_{\tau^s}(g)$ equivalent to it.

Furthermore, it is obvious that

$$B_1 = E, \tag{1}$$

where 1 is the unit element of the Weyl group.

Now we introduce a partial order in the set of elements of the Weyl group S. We say that $s_1 < s_2$ if for every root α it follows from $\alpha > 0$ and $\alpha^{s_2} > 0$ that $\alpha^{s_1} > 0$.

We show that if $s_1 < s_1 s_2$, then

$$B_{s_1 s_2} = B_{s_1} B_{s_2}. \tag{2}$$

Proof. By the definition of the operators B_s we have

$$\begin{aligned} B_{s_1} B_{s_2} f(y) &= \int\limits_{Z_A^{s_1}} \int\limits_{Z_A^{s_2}} f(y_0 s_2^{-1} z^{s_2} s_1^{-1} z^{s_1} g)\, dz^{s_2}\, dz^{s_1} \\ &= \int\limits_{Z_A^{s_1}} \int\limits_{Z_A^{s_2}} f(y_0 (s_1 s_2)^{-1} (s_1 z^{s_2} s_1^{-1}) z^{s_1} g)\, dz^{s_2} a z^{s_1}. \end{aligned} \tag{3}$$

Thus, if we can show that

$$(s_1 Z^{s_2} s_1^{-1}) \cap Z^{s_1} = E \tag{4}$$

and

$$(s_1 Z^{s_2} s_1^{-1}) Z^{s_1} = Z^{s_1 s_2}, \tag{5}$$

then, by applying Fubini's theorem to the integral (3), we find

$$B_{s_1}B_{s_2}f(y) = \int\limits_{Z_A^{s_1 s_2}} f(y_0(s_1 s_2)^{-1}z^{s_1 s_2}g)\, dz^{s_1 s_2} = B_{s_1 s_2}f(y).$$

Hence, it is sufficient to prove the relations (4) and (5).

We denote by Π_s the set of roots α belonging to Z^s, that is, those for which $\alpha > 0$ and $\alpha^s > 0$. Then (4) and (5) are equivalent to the following conditions:

$$(\Pi_{s_2})^{s_1^{-1}} \cap \Pi_{s_1} = \varphi; \quad (\Pi_{s_2})^{s_1^{-1}} \cup \Pi_{s_1} = \Pi_{s_1 s_2}. \tag{6}$$

Here $(\Pi_{s_2})^{s_1^{-1}}$ denotes the set of roots of the form $\alpha^{s_1^{-1}}$, where $\alpha \in (\Pi_{s_2}.)$ Let us prove these relations.

First, we show that $(\Pi_{s_2})^{s_1^{-1}} \cap \Pi_{s_1} = \varphi$ for any s_1, s_2. In fact, if $\alpha \in (\Pi_{s_2})^{s_1^{-1}} \cap \Pi_{s_1}$, then this means that $\alpha > 0$, $\alpha^{s_1} < 0$ and at the same time $\alpha^{s_1} > 0$, $\alpha^{s_1 s_2} < 0$, which is impossible.

Now we show that $\Pi_{s_1 s_2} \subset (\Pi_{s_1})^{s_1^{-1}} \cup \Pi_{s_2}$ for arbitrary s_1, s_2. In fact, let $\alpha \in \Pi_{s_1 s_2}$, that is, $\alpha > 0$ and $\alpha^{s_1 s_2} < 0$. Then either $\alpha^{s_1} < 0$ or $\alpha^{s_1} > 0$. In the first case $\alpha \in \Pi_{s_1}$; in the second case $\alpha^{s_1} \in \Pi_{s_2}$; and therefore, $\alpha \in (\Pi_{s_2})^{s_1^{-1}}$.

It remains to show that $\Pi_{s_1 s_2} \subset (\Pi_{s_2})^{s_1^{-1}} \subset \Pi_{s_1}$ if $s_1 < s_1 s_2$. For if $\alpha \in \Pi_{s_1}$, that is, $\alpha > 0$ and $\alpha^{s_1} < 0$, then by the condition $s_1 < s_1 s_2$ we have $\alpha^{s_1 s_2} < 0$; hence, $\alpha \in \Pi_{s_1 s_2}$. But if $\alpha \in (\Pi_{s_2})^{s_1^{-1}}$, that is, $\alpha^{s_1} > 0$, $\alpha^{s_1 s_2} < 0$, then by virtue of the condition $s_1 < s_1 s_2$ we must have $\alpha > 0$ or we would have $-\alpha > 0$, $(-\alpha)^{s_1 s_2} > 0$. But $(-\alpha)^{s_1} < 0$); consequently, $\alpha \in \Pi_{s_1 s_2}$.†

So the relation (3) for the operators B_s is proved.

We denote by s_α the element of the Weyl group corresponding to the reflection relative to the simple root α.

We show that *every element s of the Weyl group has a decomposition*

$$s = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k},$$

such that

$$B_s = B_{s_{\alpha_1}} B_{s_{\alpha_2}} \cdots B_{s_{\alpha_k}}.$$

We prove this by induction on the number of positive roots that are carried by s into negative roots. Observe that if s preserves the signs of all the roots, then $s = 1$.

Suppose that s changes the sign of precisely k positive roots. Then, among these k positive roots there must be at least one simple root α_1. It is not hard to see that $s_{\alpha_1} < s$.‡ Therefore, by the

† We use the fact that $(-\alpha)^s = -\alpha^s$.

‡ This follows immediately from the fact that s_{α_1} changes the sign only of the root α_1 and of its multiples.

relation already proved we have

$$B_s = B_{s_{\alpha_1}} B_{s_{\alpha_1}^{-1} s}.$$

The element $s_{\alpha_1}^{-1}s$ changes the sign of fewer than k positive roots, namely, of $k - n$ roots, where $n = 2$ if $2\alpha_1$ is a root and $n = 1$ otherwise. Consequently, by the inductive hypothesis, there exists a decomposition $s_{\alpha_1}^{-1}s = s_{\alpha_2} \cdots s_{\alpha_k}$ for which $B_{s_{\alpha_1}^{-1}s} = B_{s_{\alpha_2}} \cdots B_{s_{\alpha_n}}$. But then we have $B_s = B_{s_{\alpha_1}} B_{s_{\alpha_2}} \cdots B_{s_{\alpha_k}}$, as required.

5. Main Theorem on the Operators B_s. In this subsection we state the main theorem on the operators B_s. We assume further that G is a reductive group splitting over Q. (For the definition of a splitting group see p. 363.)

The main theorem states:

1. *There exist unitary operators $\bar{B}_s$ in $L_2(D_Q Z_A \setminus G_A)$ that form a representation of the Weyl group and coincide with the operators B_s on a set Φ of finite functions, everywhere dense in $L_2(\Omega)$. Here the set Φ is invariant under the operators B_s.*

This theorem is a generalization of the relation $\bar{B}^2 = 1$, which was proved in **§ 4**.

In this subsection we reduce the proof of the theorem for an arbitrary reductive group G to the proof of the following proposition.

2. *There exist unitary operators $\bar{B}_{s_0}$ in $L_2(D_Q Z_A \setminus G_A)$, where s_α is the reflection relative to the simple root α, that satisfy the relation $\bar{B}_{s_\alpha}^2 = 1$ and coincide with the operators B_{s_0} on a set Φ of finite functions everywhere dense in $L_2(\Omega)$. Here Φ can be chosen so that it is invariant under all B_{s_α}.*

LEMMA. *If 2 holds, then so does 1.*

The converse proposition is trivial.

Proof. As we have shown in **§ 5.4**, every $s \in S$ has a decomposition

$$s = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}, \tag{1}$$

such that

$$B_s = B_{s_{\alpha_1}} \cdots B_{s_{\alpha_k}}. \tag{2}$$

We set

$$\bar{B}_s = \bar{B}_{s_{\alpha_1}} \cdots \bar{B}_{s_{\alpha_k}}. \tag{3}$$

Clearly, the operators $\bar{B}_s$ so defined are unitary in $L_2(X)$ and coincide with the B_s on Φ, if Φ is chosen so that it is invariant under B_{s_α}.

It remains to show that the so-defined operators $\bar{B}_s$ form a representation of the Weyl group. With this aim we make use of the following property of the Weyl group.

The elements s_α, where α is a simple root, generate the Weyl group. A

complete system of relations between the elements s_α has the following form:

$$s_\alpha^2 = 1, \tag{4}$$

$$(s_\alpha s_\beta)^k = 1, \tag{5}$$

where $k = 2, 3, 4, 6$, *according as the angle between the vectors* α *and* β *is* $90°$, $120°$, $135°$ $150°$. To prove that the $\bar{B}_s$ form a representation of the Weyl group it is clearly sufficient to verify (5) for the corresponding operators B_{s_α}, acting on Φ. In the subsequent arguments of this subsection we assume that the operators B_{s_α} are restricted to Φ.

We investigate all possible cases separately.

1. The angle between α and β is $90°$. In this case $k = 2$, that is, we have to prove the relation

$$(B_{s_\alpha} B_{s_\beta})^2 = 1.$$

Since $B_{s_\alpha}^2 = B_{s_\beta}^2 = 1$, this relation is equivalent to

$$B_{s_\alpha} B_{s_\beta} = B_{s_\beta} B_{s_\alpha}, \tag{6}$$

which we shall now establish.

Note that $s_\alpha s_\beta = s_\beta s_\alpha$ changes the sign of the roots α and β. Hence, it follows that $s_\alpha < s_\alpha s_\beta$, $s_\beta < s_\beta s_\alpha$. Consequently, by the result obtained in **§ 5.4**, we have

$$B_{s_\alpha s_\beta} = B_{s_\alpha} B_{s_\beta} \quad \text{and} \quad B_{s_\alpha s_\beta} = B_{s_\beta s_\alpha} = B_{s_\beta} B_{s_\alpha},$$

Hence $B_{s_\alpha} B_{s_\beta} = B_{s_\beta} B_{s_\alpha}$.

2. The angle between α and β is $120°$. In this case $k = 3$, that is, we have to prove the relation

$$(B_{s_\alpha} B_{s_\beta})^3 = 1.$$

Since $B_{s_\alpha}^2 = B_{s_\beta}^2 = 1$, this relation is equivalent to

$$B_{s_\alpha} B_{s_\beta} B_{s_\alpha} = B_{s_\beta} B_{s_\alpha} B_{s_\beta}, \tag{7}$$

which we shall now prove.

It is not hard to check that $s_\alpha s_\beta s_\alpha$ carries the root α into $-\beta$, that is, into a negative root. It follows that $s_\alpha < s_\alpha s_\beta s_\alpha$, therefore,

$$B_{s_\alpha s_\beta s_\alpha} = B_{s_\alpha} B_{s_\beta s_\alpha}.$$

It is easy to check that $s_\beta s_\alpha$ carries β into the negative root $-\alpha - \beta$. Hence, $s_\beta < s_\beta s_\alpha$, and therefore,

$$B_{s_\beta s_\alpha} = B_{s_\beta} B_{s_\alpha}.$$

So we have shown that $B_{s_\alpha s_\beta s_\alpha} = B_{s_\alpha} B_{s_\beta} B_{s_\alpha}$. By similar arguments we have $B_{s_\alpha s_\beta s_\alpha} = B_{s_\beta s_\alpha s_\beta} = B_{s_\beta} B_{s_\alpha} B_{s_\beta}$; consequently,

$$B_{s_\alpha} B_{s_\beta} B_{s_\alpha} = B_{s_\beta} B_{s_\alpha} B_{s_\beta},$$

as required.

3. The angle between α and β is 135°. In this case $k = 4$ and we have to show that $(B_{s_\alpha} B_{s_\beta})^4 = 1$.

Since $B^2_{s_\alpha} = B^2_{s_\beta} = 1$, this relation is equivalent to

$$B_{s_\alpha} B_{s_\beta} B_{s_\alpha} B_{s_\beta} = B_{s_\beta} B_{s_\alpha} B_{s_\beta} B_{s_\alpha}, \tag{8}$$

which we shall now demonstrate.

In the case under discussion the lengths of the vectors α and β are in the ratio $\sqrt{2}$ to 1 (to be definite we assume that α is the root of greater length).

It is easy to verify that $s_\alpha s_\beta s_\alpha s_\beta$ carries α into $-\alpha$, hence, $s_\alpha < s_\alpha s_\beta s_\alpha s_\beta$; $s_\beta s_\alpha s_\beta$ carries β into $-\alpha - \beta$, hence, $s_\beta < s_\beta s_\alpha s_\beta$; $s_\alpha s_\beta$ carries α into $-\alpha - 2\beta$, hence, $s_\alpha < s_\alpha s_\beta$. Consequently, by the result in **§ 5.4**, we have

$$B_{s_\alpha s_\beta s_\alpha s_\beta} = B_{s_\alpha} B_{s_\beta s_\alpha s_\beta} = B_{s_\alpha} B_{s_\beta} B_{s_\alpha s_\beta} = B_{s_\alpha} B_{s_\beta} B_{s_\alpha} B_{s_\beta}.$$

Similarly, we can see that $s_\beta s_\alpha s_\beta s_\alpha$ carries β into $-\beta$, hence $s_\beta < s_\beta s_\alpha s_\beta s_\alpha$; $s_\alpha s_\beta s_\alpha$ carries α into $-\alpha - 2\beta$, hence $s_\alpha < s_\alpha s_\beta s_\alpha$; $s_\beta s_\alpha$ carries β into $-\alpha - \beta$, hence $s_\beta < s_\beta s_\alpha$. Consequently, by the result in **§ 5.4**, we have

$$B_{s_\alpha s_\beta s_\alpha s_\beta} = B_{s_\beta s_\alpha s_\beta s_\alpha} = B_{s_\beta} B_{s_\alpha s_\beta s_\alpha} = B_{s_\beta} B_{s_\alpha} B_{s_\beta s_\alpha} = B_{s_\beta} B_{s_\alpha} B_{s_\beta} B_{s_\alpha}.$$

So we have shown that

$$B_{s_\alpha} B_{s_\beta} B_{s_\alpha} B_{s_\beta} = B_{s_\beta} B_{s_\alpha} B_{s_\beta} B_{s_\alpha}.$$

4. The angle between α and β is 150°. In this case $k = 6$ and we have to show that

$$(B_{s_\alpha} B_{s_\beta})^6 = 1.$$

Since $B^2_{s_\alpha} = B^2_{s_\beta} = 1$ this relation is equivalent to

$$B_{s_\alpha} B_{s_\beta} B_{s_\alpha} B_{s_\beta} B_{s_\alpha} B_{s_\beta} = B_{s_\beta} B_{s_\alpha} B_{s_\beta} B_{s_\alpha} B_{s_\beta} B_{s_\alpha}. \tag{9}$$

In the present case the lengths of the vectors α and β are in the ratio $\sqrt{3}$ to 1.

As in the preceding cases it is easy to verify on the basis of simple geometric arguments that

$$s_\alpha < s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta,\ s_\beta < s_\beta s_\alpha s_\beta s_\alpha s_\beta,\ s_\alpha < s_\alpha s_\beta s_\alpha s_\beta,\ s_\beta < s_\beta s_\alpha s_\beta,\ s_\alpha < s_\alpha s_\beta.$$

Consequently, by the result in **§ 5.4** we have

$$B_{s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta} = B_{s_\alpha} B_{s_\beta} B_{s_\alpha} B_{s_\beta} B_{s_\alpha} B_{s_\beta}.$$

Similarly, we have

$$B_{s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta} = B_{s_\beta} B_{s_\alpha} B_{s_\beta} B_{s_\alpha} B_{s_\beta} B_{s_\alpha},$$

and (9) follows immediately from this. The proof of the lemma is now complete.

6. Reduction to Rank 1. Here we show that the proof of Theorem 2 can be reduced to the case of groups of rank 1.†

Let G be an arbitrary reductive group, and $\mathfrak{G}$ its Lie algebra. Throughout this subsection we use the notation introduced in **§ 5.1**.

We denote by ${}^\alpha\mathfrak{Z}$, where α is a simple root, the minimal subalgebra of $\mathfrak{G}$ generated by the root spaces $\mathfrak{G}_\beta$, where β ranges over all positive roots and all the roots proportional to α. Next we denote by $\mathfrak{G}^\alpha$ the minimal subalgebra of $\mathfrak{G}$ generated by the root spaces $\mathfrak{G}_\beta$, where β ranges over the roots proportional to α.

Let ${}^\alpha Z$ and G^α be the groups corresponding to these subalgebras. By definition, G^α is a simple algebraic group of rank 1. Therefore, its Weyl groups consists of two elements.

Let Z^α be a maximal unipotent subgroup of G^α; obviously, the Lie algebra of Z^α is generated by the roots spaces $\mathfrak{G}^\alpha$, where β ranges over the positive roots that are multiples of α. We denote by B the Weyl operator in the space $D^\alpha_Q Z^\alpha_A \setminus G^\alpha_A$.

The object of this subsection is to prove that *if Theorem 2 holds for the operator B in the space $L_2(D^\alpha_Q Z^\alpha_A \setminus G^\alpha_A)$, then it also holds for the operator B_s in the space $L_2(D_Q Z_A \setminus G_A)$.*

To begin with we prove that $Z_A \setminus G_A$ is a fiber space whose base is the space ${}^\alpha Z_A \setminus G_A$, while the fiber is the space $Z^\alpha_A \setminus G^\alpha_A$.

For since $Z^\alpha_A \supset Z_A$, there is a natural map

$$Z_A \setminus G_A \to {}^\alpha Z_A \setminus G_A,$$

by which $Z_A \setminus G_A$ is equipped with the structure of a fiber space with the base ${}^\alpha Z_A \setminus G_A$ and the fiber $Z_A \setminus {}^\alpha Z_A$. It remains to establish the isomorphism

$$Z_A \setminus {}^\alpha Z_A \cong Z^\alpha_A \setminus G^\alpha_A.$$

We denote by $\tilde{Z}^\alpha$ the subgroup of Z complementary to Z^α, that is, the subgroup whose algebra is generated by the root spaces $\mathfrak{G}_\beta$, where β ranges over all positive roots that are not multiples of α. The group Z_A splits into the semidirect product

$$Z_A = \tilde{Z}^\alpha_A Z^\alpha_A. \tag{1}$$

On the other hand, from the definition of ${}^\alpha Z_A$ it follows that it also splits to a semidirect product

$${}^\alpha Z_A = \tilde{Z}^\alpha_A G^\alpha_A. \tag{2}$$

From (1) and (2) it follows immediately that

$$Z_A \setminus {}^\alpha Z_A \cong Z^\alpha_A \setminus G^\alpha_A.$$

† The rank of a group is the number of its simple roots.

We now observe that in the formula for the operator B_s:

$$B_{s_\alpha}\varphi(y) = \int\limits_{Z_A^Q \backslash Z_A^\alpha} \varphi(y_0 s_\alpha^{-1} z^\alpha g)\, dz^\alpha \tag{3}$$

the integration is taken over the set of points $y_0 s_\alpha^{-1} z^\alpha g$ that belong to the same fiber of the fiber space $Z_A \setminus G_A$ as y itself. For since $s_\alpha \in {}^\alpha Z_A$, under the map $G_A \to {}^\alpha Z_A \setminus G_A$ the set $y_0 s_\alpha^{-1} Z_A^\alpha g$ goes into a single point, the same into which $y = y_0 g$ goes.

We assume that the function $\varphi(y)$ is concentrated in a sufficiently small domain. Then we can introduce a local system of coordinates (u, v) in this domain, where v gives a point of the base ${}^\alpha Z_A \setminus G_A$, and u a point of the fiber $Z_A^\alpha \setminus G_A^\alpha$.

From the remark above it follows that B_s acts on $\varphi(y) = \varphi(u, v)$, as if it were a function of the variable $u \in Z_A^\alpha \setminus G_A^\alpha$ only.

Also, B_s can be expressed in terms of the operator of horospherical automorphism B, acting in the space of functions $\varphi(u)$. Specifically,

$$B_{s_\alpha}\varphi(u, v) = B\varphi(u, v).$$

Consequently, if it is proved that $B^2 = 1$ on an everywhere dense set Φ_0 of functions on $Z_A^\alpha \setminus G_A^\alpha$ invariant relative to D_Q^α, then it follows that $B_{s_\alpha}^2 = 1$ on the set Φ_α of all functions on $Z_A \setminus G_A$ which as functions on the fibers belong to Φ_0. It is not hard to verify that the intersection over all α of the so-defined sets Φ_α contains the set Φ of functions invariant relative to D_Q, which is everywhere dense in $L_2(D_Q Z_A \setminus G_A)$. So we have shown that the verification of Theorem 2 reduces to the discussion of groups of rank 1.

Hence the proof of Theorem 1 for a reductive group G reduces to the proof of Theorem 2 for its semisimple subgroups of rank 1.

Using this reduction we can prove Theorem 1 for a wide class of reductive groups. The problem whether or not Theorem 1 is true for all reductive groups awaits solution.

Theorem 1 is true for any reductive group G splitting over Q.

(The definition of a splitting group was given on p. 363.)

For let G^α be the subgroup defined on p. 376. It is not hard to check that in this case G^α is a group of unimodular matrices of order 2 over Q. Thus, the proof of Theorem 1 for G reduces to that of 2 for the group of unimodular matrices of order 2 over Q. But for the group of unimodular matrices of order 2 over Q this Theorem 2 was already proved in **§ 4**.

So the main theorem is proved for all splitting reductive groups. For arbitrary reductive groups over Q, it reduces to groups of rank 1.

§ 6. THE REPRESENTATIONS GENERATED BY THE HOMOGENEOUS SPACE $G_Q \setminus G_A$

1. The Homogeneous Space $G_Q \setminus G_A$. Let G be a linear algebraic group defined over Q, G_A its group of adeles, and G_Q the group of principal adeles.

G_Q is a discrete subgroup of G_A. The proof of this proposition is exactly the same as in the case of unimodular matrices of order 2 (see **§ 4**).

We examine the space $X = G_Q \setminus G_A$. Since G_Q is a discrete group, the space X is locally isomorphic to G_A. Consequently, the right-invariant measure on G_A induces a measure on the base $X = G_Q \setminus G_A$ that is invariant under the translation of G_A.

The following fundamental theorem of Borel [6] tells us when the measure of the space X is finite.

THEOREM 1. *The measure of the space X is finite if and only if G contains no nontrivial characters defined over Q, that is, morphisms of G into Q.*

For example, the factor space of the group of ideles by the principal ideles has infinite measure, but the factor space of the group of adeles by the principal adeles has finite measure.

From Theorem 1 it also follows that *if G is a semisimple or unipotent group, then the factor space $X = G_Q \setminus G_A$ has finite measure.* Now we mention that on G_A, and consequently also on X, there is a canonical way of normalizing the measure.† Following Weil, we call the so determined number for the measure of the space X the Tamagawa number of G. It is a very interesting arithmetic characteristic of G.

The main object of study in the present section is the representation generated by the homogeneous space $X = G_Q \setminus G_A$. The structure of the decomposition of this space into irreducible subspaces is closely connected with arithmetical properties of G. At the present time the complete description of the decomposition of this representation into irreducible subspaces is not known. There is good reason to hope that once it has been found, it will shed light on many number-theoretic problems (see, for example, **§ 4** Appendix II).

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† This method simply consists in the following. Let ω be the differential form defining the measure on G. As is well known, it may be assumed that it is defined over Q, and so has the form $\varphi(x_1, \ldots, x_m)\, dx_1 \wedge \cdots \wedge dx_m$, where $x_1, \ldots, x_m$ are local coordinates, and φ is a rational function with coefficients in Q. This form is uniquely determined to within multiplication by a rational number. The ω form induces uniquely determined measures on the groups $G_p (p = \infty, 2, \ldots)$ and so, a certain measure on G_A. This measure is uniquely determined, because if we take instead of ω the form $\omega' = \lambda\omega$, $\lambda \in Q$, then the measure is multiplied by a certain power of the norm $|\lambda|$ of the idele λ. Since $\lambda \in Q$, we have $|\lambda| = 1$ (see **§ 1**).

The present section essentially deals with the separation of the discrete part of the spectrum of the representation from the continuous part of the spectrum of maximal dimension.

We now discuss the general situation when G has nontrivial characters defined over Q. Every character χ of G induces a morphism χ_A of G_A into the group of ideles A^*. We denote by G_A^0 the subgroup of all $g \in G_A$ such that the norm $|\chi_A(g)|$ of the idele $\chi_A(g)$ is equal to 1 every character χ of G.

It is not hard to see that $G_Q \subset G_A^0$. For if $g \in G_Q$, then $\chi_A(g)$ is a principal idele, and therefore $|\chi_A(g)| = 1$ (see **§ 1**). The following theorem is due to Borel [6].

THEOREM 2. *The factor space $X^0 = G_Q \setminus G_Q \setminus G_A^0$ has finite volume.*

In many important cases the space $X^0 = G_Q \setminus G_A^0$ turns out to be compact. Necessary and sufficient conditions for the compactness of X^0 were proposed by Godement as a conjecture and have recently been proved by Borel and independently by Mostow and Tamagawa.

THEOREM 3. *The space $X^0 = G_Q \setminus G_A^0$ is compact if and only if all unipotent elements of G_Q belong to its radical; in particular, if G is reductive, then X^0 is compact if and only if G_Q has no unipotent elements.*

From Theorem 3 it follows that if G is unipotent, then $X = G_Q \setminus G_A$ is compact. For if G is unipotent, then $X^0 = G_Q \setminus G_A^0$ is compact by Theorem 3. Furthermore, if G is unipotent, then it has no nontrivial characters defined over Q, and hence $G_A^0 = G_A$.

2. Investigation of the Spectrum of the Representation for a Compact Space $G_Q \setminus G_A/K_A$. In this subsection we discuss the simplest case when all the elements of G_Q are semisimple. By Theorem 3 in **§ 6.1** the space $X^0 = G_Q \setminus G_A^0$ is then compact.

If G has no nontrivial characters defined over Q, then $G_A^0 = G_A$; consequently, $X = G_Q \setminus G_A$ is compact. Therefore, the representation in $L_2(X)$ splits into the direct sum of countably many irreducible representations (see Chapter I, **§ 2**).

We now study the general case when G has nontrivial characters.

Let K be the center of G, K_A and K_Q the corresponding groups of adeles and principal adeles. We state without proof the following property of reductive groups:

The subgroup $G_A^0 K_A$ has finite index in G_A.

From this property and the compactness of $G_Q \setminus G_A^0$ it follows that the space of double cosets $G_Q \setminus G_A/K_A$ is also compact.

On the basis of this fact we split the space H into the direct integral of invariant spaces H_π, each of which has a discrete spectrum.

Let $\pi(k)$ be any unitary character on K_A that is identically equal to the unit element on K_Q. We denote by H_π the space of

functions $f_\pi(g)$ on G_A that satisfy the following conditions:

1. $f_\pi(gk) = f_\pi(g)\pi(k)$ for every $k \in K_A$,
2. $f_\pi(\gamma g) = f_\pi(g)$ for every $\gamma \in G_Q$,
3. $\int\limits_{G_Q \backslash G_A / K_A} |f(g)\, dg| < \infty.$

H splits into the continuous direct sum of the spaces H_π according to the following formulae:

$$f(g) = \int f_\pi(g)\, dg,$$

$$\int\limits_{G_Q \backslash G_A} |f(g)|^2\, dg = \int\limits_{\Pi} \int\limits_{G_Q \backslash G_Q / K_A} |f_\pi(g)|^2\, dg\, d\pi,$$

where $f_\pi(g)$ is the component of the vector $f(g) \in H$ in H_π that is defined by the formula:

$$f_\pi(g) = \int\limits_{K_A / K_Q} f(gk)\bar{\pi}(k)\, dk.$$

By the same method as in Chapter I, **§ 2**, it can be shown that H_π is itself a sum of countably many irreducible representations of G_A.

In fact, it can be shown the trace of the operator

$$T_\varphi = \int\limits_{G_A} \varphi(g) T(g)\, dg$$

in H_π is finite, where φ is an arbitrary Schwartz-Bruhat function. Hence, it follows that for every irreducible unitary representation in the representation generated by $G_Q \backslash G_A$ the trace of T_φ is also finite, where φ is a Schwartz-Bruhat function on G_A. But, as was proved in **§ 3.5**, if the trace of the operator T_φ of an irreducible representation of G_A is finite, then this irreducible representation splits into the tensor product of irreducible representations T_p of the groups G_p, and all but a finite number of the representations T_p contain precisely one linearly independent vector invariant under the group U_p.

Thus, every irreducible representation of G_A occurring in the decomposition of the representation generated by $G_Q \backslash G_A$ is a tensor product of irreducible representations T_p of the groups G_p; and all but a finite number of the representations T_p have precisely one linearly independent vector invariant under the subgroup U_p of integral p-adic matrices.

It is not known whether all irreducible representations of G_A have this property. There is good reason to conjecture that the answer to this question is affirmative.

Nor is it known whether this property holds for the irreducible representations of G_A that occur in the representation generated by the homogeneous space $X = G_Q \setminus G_A$, in case $G_Q \setminus G_A/K_A$ is not compact. For those subspaces of $L_2(X)$ that are studied in the present section the answer is affirmative.

3. The Space of Horospheres. Let G be an algebraic reductive group defined over Q and such that the space $G_Q \setminus G_A/K_A$ is compact.

According to Theorem 3 in **§ 6.1**, in this case G has unipotent elements. We denote a maximal unipotent subgroup of G by Z. We define *horospheres* in the space $X = G_Q \setminus G_A$ as the images of the cosets $Z_A g$ under the natural map†

$$G_A \to G_Q \setminus G_A.$$

Thus, every horosphere in X is a set of points of the form

$$x_z = x_0 z g,$$

where x_0 is the point in X corresponding to the unit coset of G_A, g is any fixed element in G_A, and z ranges over the subgroup Z_A. Obviously, this set is isomorphic to $Z_Q \setminus Z_A$ and consequently, compact.

Since translations from G_A carry horospheres into horospheres, the set of horospheres is a homogeneous space of G_A, which we denote by Ω.

Let us find the stability subgroup of Ω.

Let G' be the normalizer of Z in G. We know that G' splits into the semidirect product

$$G' = DZ$$

of its normal subgroup Z and a certain reductive subgroup D. We recall that all the elements of D are semisimple.

We show that the stability group of Ω is $D_Q Z_A$.

Indeed, by arguments similar to those in **§ 4.3** it is not hard to verify that this stability subgroup is generated by the subgroups Z_A and $G'_A \cap G_Q$. Since $G'_A = D_A Z_A$, we have $G'_A \cap G_Q = D_Q Z_Q$. Clearly, the subgroup generated by Z_A and $D_Q Z_Q$ is $D_Q Z_A$.

So we have shown that

$$\Omega = D_Q Z_A \setminus G_A.$$

† A more general definition of horospheres will be given in § 7.

Apart from the homogeneous space Ω we also introduce the homogeneous space

$$Y = Z_A \setminus G_A.$$

We call Ω the space of horospheres, and Y the underlying affine space of the group G_A.

4. The Horospherical Map and the Operator M. We consider the space $L_2(X)$, $X = G_Q \setminus G_A$. With every function $f(x)L_2(x)$ we associate its integral over the horospheres:

$$\varphi(g) = \int\limits_{Z_Q \setminus Z_A} f(x_0 zg)\, dz, \tag{1}$$

where x_0 is the point in X corresponding to the unit coset. The correspondence

$$f(x) \to \varphi(g)$$

is called the *horospherical map.*

Obviously $\varphi(g)$ satisfies the following condition for any $\delta \in D_Q$ and $z \in Z_A$:

$$\varphi(\delta zg) = \varphi(g).$$

So we can regard it as a function in the space of horospheres $\Omega = D_Q Z_A \setminus G_A$ and write $\varphi(y), y \in \Omega$, instead of $\varphi(g)$.

Let H^0 be the kernel, and H' the image of $L_2(X)$, under the horospherical map. We endow H' with the structure of a Hilbert space, by setting

$$H' = L_2(X)/H^0.$$

The scalar product in H' is noted by $[\psi_1, \psi_2]$.

By analogy with the case of the group of unimodular matrices of order 2, which was discussed in **§ 4**, we introduce an operator M.

Let $\psi(y)$ be an arbitrary continuous finite function on Ω. It gives rise to a functional in H' according to the formula

$$(\psi, \varphi) = \int\limits_{\Omega} \varphi(y)\overline{\psi(y)}\, dy. \tag{2}$$

As in **§ 4.10**, it is easy to verify the following estimate:

$$\int\limits_K |\varphi(y)|\, dy < c(K)[\varphi, \varphi]^{1/2},$$

where K is any compactum in Ω and $c(K)$ a constant. From this estimate, it follows immediately that (ψ, φ) is a linear continuous functional in H'. Consequently, by Riesz' theorem,

$$\int\limits_{\Omega} \varphi(y)\overline{\psi(y)}\, dy = [\varphi, M\psi], \tag{3}$$

where $M\psi \in H'$, and brackets denote, of course, the scalar product in H'.

We take (3) as the definition of the operator M, which therefore carries continuous finite functions on Ω into functions in H'.

The following properties of M can be established just as in **§ 4.10**.

1. M commutes with the representation operators in Ω, that is

$$M(\varphi(yg)) = (M\varphi)(yg).$$

2. $(M\varphi, \varphi) \geq 0$ for every finite continuous function (y).

3. The set of functions of the form M, where φ ranges over the continuous finite functions on Ω, is everywhere dense in H'.

5. An Explicit Expression for the Operator M. Here we show that the operator M can be given by the following formula:

$$M = \sum_s B_s, \tag{1}$$

where B_s is the Weyl operator defined in **§ 5.3**, and the summation is taken over all elements of the Weyl group.

The derivation of this formula proceeds just as that of the analogous formula in **§ 4.11** for the case of the group of the matrices of order 2.

By exactly the same arguments as in **§ 4.11** we obtain the following expression for M:

$$M\psi(y) = \int_{Z_Q \backslash Z_A} \sum_{\gamma \in D_Q Z_Q \backslash G_Q}' \psi(y_0 \gamma z)\, dz; \tag{2}$$

In Σ' precisely one representative is taken from every coset $D_Q Z_Q \gamma$; g is an arbitrary element of the coset $D_Q Z_Q \setminus G_A$ corresponding to y.

Now we transform formula (2). For this purpose we choose in every coset $D_Q Z_Q \gamma$ a canonical representative. Here we make use of the following fact. Let N_Q be the normalizer of D_Q in G_Q. Then every element $\gamma \in G_Q$ can be written as a product

$$\gamma = znz', \tag{3}$$

where $n \in N_Q$, z, $z' \in Z_Q$. We choose one representative s in every coset of N_Q/D_Q. Then the decomposition (3) takes the form

$$\gamma = z\delta s z', \tag{4}$$

where z, $z' \in Z_Q$; $\delta \in D'_Q$. Here the element s, that is, the coset of N_Q/D_Q, is uniquely determined by γ.

Hence, every coset of $D_Q Z_Q \setminus G_Q$ contains an element of the form sz, $z \in Z_Q$, and s is uniquely determined once the coset is given.

It is not hard to verify that the elements sz and sz' belong to one

and the same coset $D_Q Z_Q \setminus G_Q$ if and only if $zz'^{-1} \in Z^s_Q$, where $Z^s_Q = s^{-1} Z_Q s \cap Z_Q$.

Hence, the expression (2) can be rewritten in the following form:

$$M\varphi(y) = \sum_s \int_{Z_Q \setminus Z_A} \Big(\sum_{z' \in Z^s_Q \setminus Z_Q} \varphi(y_0 s z' z g) \, dz \Big)$$

$$= \sum_s \int_{Z^s_Q \setminus Z_A} \varphi(y_0 s z g) \, dz. \tag{5}$$

In (5) we split the integration over $Z^s_Q \setminus Z_A$ into integration over $Z^s_Q \setminus Z^s_A$, where $Z^s_A = s^{-1} Z^s_A \cap Z_A$, and integration over $Z^s_A \setminus Z_A$. Hence, we find that

$$\int_{Z^s_Q \setminus Z_A} \varphi(y_0 s z g) \, dz = \int_{Z^s_A \setminus Z_A} \int_{Z^s_Q \setminus Z^s_A} \varphi(y_0 s z' z g) \, dz' \, dz.$$

Since $\varphi(y_0 s z' z g) = \varphi(y_0 s z g)$ for every $z' Z^s_A$, that is, the expression under the integral sign does not depend on z', we have

$$\int_{Z^s_Q \setminus Z_A} \varphi(y_0 s z g) \, dz = \int_{Z^s_Q \setminus Z_A} \varphi(y_0 s z g) \, dz = B_{s^{-1}} \varphi,$$

where $B_{s^{-1}}$ is the Weyl operator (see **§ 5.3**). Thus, formula (5) takes the form

$$M\varphi = \Big(\sum_s B_s \Big) \varphi,$$

that is, $M = \sum_s B_s$, as required.

6. Structure of the Space H'. In this subsection we investigate the structure of H', by using the expression for M in terms of the operators B_s obtained in **§ 6.5**. We recall that H' denotes the image of $L_2(X)$ under the horospherical map (see **§ 6.4**). We also assume that the group G splits (see the definition on p. 363). Then the following Theorem 1 holds: *in the space $L_2(D_Q Z_A \setminus G_A)$ there exist unitary operators $\bar{B}_s$ that form a representation of the Weyl group S and coincide with the operators B_s on a certain set Φ of functions*, everywhere dense in $L_2(D_Q Z_A \setminus G_A)$ and invariant under the operators B_s.

We show that *if the group G splits, then the space*

$$H_0 = L_2(\Omega) \cap H'$$

decomposes into the same irreducible representations as $L_2(\Omega)$ but, in contrast to the latter, every irreducible representation occurs in the decomposition with multiplicity 1.

Proof. We denote by H_0 the closure in $L_2(\Omega)$ of the set of functions of the form $\bar{M}\varphi$, $\bar{M} = \Sigma \bar{B}_s$, where $\varphi \in \Phi$ (Φ is defined in the statement of Theorem 1). It is not hard to check that H_0 coincides with the set of all $f \in L_2(\Omega)$ such that $\bar{B}_s f = f$ for all $s \in S$.

Now we observe as, was proved in **§ 5.2**, that the multiplicity with which a given irreducible representation occurs in $L_2(\Omega)$ is equal to the order of the Weyl group. Moreover, the operators B_s carry each irreducible representation into an equivalent representation.

We consider the sum H^r of all irreducible subspaces contained in $L_2(\Omega)$ and equivalent to the given irreducible subspace. By what we have said, each of the operators $\bar{B}_s$ carries H^r into itself and is given in H^r by a matrix whose order is equal to the order of the Weyl group. These matrices form the regular representation of the Weyl group (under the condition that we only consider irreducible representations of G_A in general position).

Clearly, the unit representation of the Weyl group S acts in the space of the functions $\bar{M}\psi$, $\psi \in H^r$. Therefore, the question of the multiplicity with which a given irreducible representation occurs in H_0 reduces to the question of the multiplicity to which the unit representation of the finite group S occurs in the regular representation of this group. It is well known that this multiplicity is 1. So we have shown that all irreducible representations of G_A that are contained in $L_2(\Omega)$ occur in H_0, and each only once.

Now we show that $H_0 \subset H'$. As in the preceding subsection we denote by $(\ , \ )$ the scalar product in $L_2(\Omega)$, and by $[\ , \ ]$ the scalar product in H'. Let $f \in H_0$; we show that $f \in H'$. From the fact that $f \in H_0$ it follows that there exists a sequence of functions $\varphi_n \in \Phi$ such that

$$(f - M\varphi_n, f - M\varphi_n) \to 0, \quad (\varphi_n, \varphi_n) < C. \tag{1}$$

From (1) it follows that

$$[M\varphi_n, M\varphi_n] = (M\varphi_n, \varphi_n) < C_1. \tag{2}$$

Consequently, we can choose from the sequence $M\varphi_n$ a subsequence that is weakly convergent in the sense of H' to a function $f_1 \in H'$.

Without loss of generality we may assume that the sequence $M\varphi_n$ itself converges weakly to f_1. Now we show that $f_1 = f$. We use the fact that f_1 and f are measurable functions, summable over every compactum in Ω. Therefore, it is sufficient to show that

$$(f_1, \varphi) = (f, \varphi) \tag{3}$$

for every finite continuous function φ. We have

$$(f_1, \varphi) = [f_1, M\varphi] = \lim_{h\to\infty} [M\varphi_n, M\varphi]$$
$$= \lim_{h\to\infty} (M\varphi_n, \varphi) = (f, \varphi), \tag{4}$$

because

$$(M\varphi_n - f, M\varphi_n - f) \to 0.$$

So we have shown that

$$H_0 \subset H'.$$

Finally, we show that the intersection of the orthogonal complement H_1 to H_0 in $L_2(\Omega)$ is trivial. For suppose that $f \in H_1 \cap L_2(\Omega)$. From the fact that $f \in H_1$ it follows that $[f, M\varphi] = 0$ for every function $\varphi \in \Phi$. Consequently,

$$(f, \varphi) = 0 \quad \text{for every } \varphi \in \Phi. \tag{5}$$

Since Φ is everywhere dense in $L_2(\Omega)$, we see that $f \in L_2(\Omega)$ implies that $f = 0$.

From what we have shown it follows that $H_0 = L_2(\Omega) \cap H'$.

§ 7. DISCRETENESS OF THE SPECTRUM

1. Horospheres in the Space $X = G_Q \backslash G_A$. Let Z be a maximal connected unipotent subgroup of the reductive group G, and T a maximal torus of G contained in the normalizer of Z and splitting over Q. We denote by $\mathfrak{Z}$ and $\mathfrak{T}$ the Lie algebras of the groups Z and T. The space $\mathfrak{Z}$ can be represented as the sum of root spaces $\mathfrak{Z}_\alpha$:

$$\mathfrak{Z} = \sum_\alpha \mathfrak{Z}_\alpha,$$

where α is a linear form on $\mathfrak{T}$.

Let $\alpha_1, \ldots, \alpha_n$ be the simple roots and Π a subset of the set of simple roots. We denote by $\mathfrak{Z}^\Pi$ the union of all the root spaces $\mathfrak{Z}_\alpha$ corresponding to all positive roots $\alpha = \Sigma c_k \alpha_k$ for which $\sum_{\alpha_k \in \Pi} c_k > 0$. Clearly, $\mathfrak{Z}^\Pi$ is a subalgebra of $\mathfrak{Z}$.

We denote by Z^Π the subgroup corresponding to $\mathfrak{Z}^\Pi$. We call Z^Π, and also every subgroup conjugate to Z^Π under an element of G_Q, a *horospherical subgroup*.

For example, let G be the group of all nonsingular matrices of order n. It is not hard to show that every horospherical subgroup of G is conjugate to the subgroup of all block triangular matrices of the

following form:

$$\begin{pmatrix} E_{k_1} & & & * \\ & E_{k_2} & & \\ & & \ddots & \\ 0 & & & E_{k_s} \end{pmatrix}.$$

Here E_{k_i} denotes the unit matrix of order k_i; k_i are fixed natural numbers such that $k_1 + \cdots k_s = n$; arbitrary elements stand above the diagonal, and zeros below.

The images of the cosets $Z_A^{\Pi} g$ under the natural map

$$G_A \to X = G_Q \setminus G_A$$

are called Π-*horospheres* in X, or simple horospheres. Thus, every horosphere in X is a set of points of the form $x_0 z_A^{\Pi} g$, where x_0 is the point corresponding to the unit coset, g any fixed element in G_A, and z_A^{Π} ranges over the subgroup Z_A^{Π}.

Observe that Π-*horospheres are compact sets*, because the set of points of a Π-horosphere is homeomorphic to the factor space $(G_Q \cap Z_A^{\Pi}) \setminus Z_Q^{\Pi} = Z_Q^{\Pi} \setminus Z_A^{\Pi}$. This factor space is compact, because Z_A^{Π} is unipotent (see **§ 6.1**).

From the definition it is obvious that the set of all Π-horospheres for a given Π is a homogeneous space of G_A. Let us find the stability group of this space.

First, we find the normalizer N^{Π} of Z^{Π} in G. We show presently that N^{Π} is a semidirect product of a reductive group G^{Π} and of the group Z^{Π}.

As we know (see **§ 5.1**), the Lie algebra of G has the following form:

$$\mathfrak{G} = \mathfrak{T} + \mathfrak{C} + \sum \mathfrak{G}_{\alpha}.$$

By definition,

$$\mathfrak{Z}^{\Pi} = \sum_{\alpha \in \Sigma_{\Pi}} \mathfrak{G}_{\alpha},$$

where α ranges over the set Σ_{Π} of all positive roots $\alpha = \sum c_k \alpha_k$, in which $c_k > 0$ for at least one $\alpha_k \in \Pi$.

Let us find the normalizer $\mathfrak{N}^{\Pi}$ of the algebra $\mathfrak{Z}^{\Pi}$. We denote by $\overline{\Pi}$ the set of simple roots that do not belong to Π. We show that

$$\mathfrak{N}^{\Pi} = \mathfrak{T} + \mathfrak{C} + \sum_{(\overline{\Pi})} \mathfrak{G}_{\alpha} + \mathfrak{Z}^{\Pi},$$

where the sum is taken over all roots α that are linear combinations of roots in $\overline{\Pi}$.

First of all, it is clear that $\mathfrak{T} + \mathfrak{C} \subset \mathfrak{N}^{\Pi}$, because

$$[\mathfrak{T} + \mathfrak{C}, \mathfrak{G}_{\alpha}] \subset \mathfrak{G}_{\alpha}$$

for every root α.

Now we show that if a root β is a linear combination of roots in $\overline{\Pi}$, then $\mathfrak{G}_\beta \subset \mathfrak{N}^\Pi$. For from the definition of the set Σ_Π it follows immediately that if $\alpha \in \Sigma_\Pi$ and $\alpha + \beta$ is a root, then $\alpha + \tau \in \Sigma_\Pi$. Consequently, since $[\mathfrak{G}_\beta, \mathfrak{G}_\alpha] = 0$ when $\alpha + \beta$ is not a root, and $[\mathfrak{G}_\beta, \mathfrak{G}_\alpha] = \mathfrak{G}_{\alpha+\beta}$ when $\alpha + \beta$ is a root, we have $[\mathfrak{G}_\beta, \mathfrak{Z}^\Pi] \subset \mathfrak{Z}^\Pi$, hence, $\mathfrak{G}_\beta \subset \mathfrak{N}^\Pi$.

So we have established that $\mathfrak{N}^\Pi \supset \mathfrak{T} + \mathfrak{C} + \sum_{(\overline{\Pi})} \mathfrak{G}_\alpha + \mathfrak{Z}^\Pi$. We show that we have, in fact, equality. From $\mathfrak{T} \subset \mathfrak{N}^\Pi$ it follows that

$$\mathfrak{N}^\Pi = \mathfrak{T} + \mathfrak{C} + \sum_\alpha (\mathfrak{N}^\Pi \cap \mathfrak{G}_\alpha).$$

We assume that for some root β that does not belong to Σ_π and is a linear combination of roots from $\overline{\Pi}$ the intersection $\mathfrak{N}^\Pi \cap \mathfrak{G}_\beta$ is not empty. Let $\mathfrak{g}_\beta \neq 0$ be an element in this intersection.

Since every positive root either belongs to Σ_π, or is a linear combination of roots in $\overline{\Pi}$, we have $\beta < 0$. Consequently, $-\beta < 0$, and hence $-\beta \in \Sigma_\pi$, $\mathfrak{G}_{-\beta} \subset \mathfrak{Z}^\Pi$.

We know that $[\mathfrak{g}_\beta, \mathfrak{G}_{-\beta}] \neq 0$. On the other hand, since $[\mathfrak{g}_\beta, \mathfrak{G}_{-\beta}] \subset \mathfrak{G}$, the set $[\mathfrak{g}_\beta, \mathfrak{G}_{-\beta}]$ is not contained in $\mathfrak{Z}^\Pi$. This shows that $\mathfrak{g}_\beta$ does not belong to the normalizer $\mathfrak{N}^\Pi$ of $\mathfrak{Z}^\Pi$, in contradiction to our assumption.

So we have shown that

$$\mathfrak{N}^\Pi = \mathfrak{T} + \mathfrak{C} + \sum_{(\overline{\Pi})} \mathfrak{G}_\alpha + \mathfrak{Z}^\Pi, \tag{1}$$

where the sum is taken over all roots that are representable as linear combinations of roots in $\overline{\Pi}$. We set

$$\mathfrak{G}^\Pi = \mathfrak{T} + \mathfrak{C} + \sum_{(\overline{\Pi})} \mathfrak{G}_\alpha.$$

Then (1) means that $\mathfrak{N}^\Pi$ is a direct sum

$$\mathfrak{N}^\Pi = \mathfrak{G}^\Pi + \mathfrak{Z}^\Pi.$$

It is not hard to verify that $\mathfrak{G}^\Pi$ is a reductive algebra, and that the system of its simple roots coincides with $\overline{\Pi}$.

Going over from Lie algebras to groups we conclude that the normalizer N^Π of Z^Π is a semidirect product

$$N^\Pi = G^\Pi Z^\Pi, \tag{2}$$

where G^Π is a reductive group whose Lie algebra is $\mathfrak{G}^\Pi$.

Now we look for the stability group in the space of Π-horospheres. It is not difficult to verify that this stability group is generated by the subgroups Z_A^Π and N_Q^Π. Since $N_Q^\Pi = G_Q^\Pi Z_Q^\Pi$, by virtue of (2), we conclude:

The stability group in the space Ω^Π of the Π-horospheres is the

group $G_Q^{\Pi} Z_A^{\Pi}$. Thus,

$$\Omega^{\Pi} = G_Q^{\Pi} Z_A^{\Pi} \setminus G_A.$$

2. Statement of the Main Theorem. We denote by $H^0(G_Q \setminus G_A)$ the intersection of the kernels of all horospherical maps, that is, the set of all functions $f(x) \in L_2(G_Q \setminus G_A)$ whose integrals over all horospheres are zero.

In other words, $H^0(G_Q \setminus G_A)$ consists of all functions $f(g)$ on G_A that satisfy the following conditions:

1. $f(\gamma g) = f(g)$ for every $\gamma \in G_Q$;
2. $\displaystyle\int_{G_Q \setminus G_A} |f(g)|^2 \, dg < \infty$;
3. $\displaystyle\int_{Z_Q^{\Pi} \setminus Z_A^{\Pi}} f(zg) \, dz = 0$ for every $g \in G_A$ and every subset Π of simple roots.

We recall that all horospheres are compact; thus, the integration in 3 is over a compact set.

Clearly, the space H^0 is invariant under the representation operators $T(g)$ of G_A:

$$T(g_0) f(g) = f(g g_0).$$

The main task of this section is the decomposition of the representation in H^0 into irreducible representations. In this subsection we give a statement of the main result.

First, we decompose H^0 into subspaces H_π^0. Let K be the center of G. Obviously, the space H^0 is invariant under the transformations

$$f(g) \to f(kg), \quad k \in K_A$$

and these transformations commute with the operators $T(g)$. Furthermore, $f(kg) = f(g)$ for every $k \in K_Q$.

Hence, it follows that H^0 can be decomposed into the continuous direct sum of the spaces H_π^0;

$$H^0 = \int H_\pi^0 \, d\pi,$$

where π ranges over the set of unitary characters of K_A that are identically equal to unity on K_Q. The space H_π^0 consists of the functions $f(g)$ on G_A that satisfy the following conditions:

1. $f(\gamma g) = f(g)$ for every $\gamma \in G_Q$;
2. $f(kg) = \pi(k) f(g)$ for every $k \in K_A$;
3. $\displaystyle\int_{G_Q \setminus G_A / K_A} |f(g)|^2 \, dg < \infty$;

$\displaystyle\int_{Z_Q^{\Pi} \setminus Z_A^{\Pi}} f(zg) \, dz = 0$ for every $g \in G_A$ and every subset Π of simple roots.

Thus, the task of decomposing the representation in H^0 into irreducible representations reduces to the same task in the spaces H^0.

The main theorem of this section asserts that H^0 splits into the direct sum of a countable number of invariant irreducible subspaces.

In fact, we shall prove even more, namely that for every positive definite Schwartz-Bruhat functions $\gamma(g)$ on G_A the operator T_φ has a trace in H^0_π. From this result it follows (see **§ 3**) that every irreducible unitary representation of G_A occurring in H^0 is a tensor product of irreducible unitary representations of the groups G_p.

3. Siegel Sets on G_A. We denote by Z_∞ the subgroup of G_A consisting of the elements of the form

$$\tilde{z} = (z_\infty, 1, \dots). \tag{1}$$

Similarly, we denote by T_∞ the group of elements of the form

$$\tilde{t} = (t_\infty, 1, \dots). \tag{2}$$

We use the term *Siegel set connected with the subgroup* Z for a subset of G_A of the form

$$Z_\infty \tilde{T}_\infty V, \tag{3}$$

where V is a compact set and $\tilde{T}_\infty$ a semibounded subset of T_∞, that is, a subset such that $t^{-1}zt$ is bounded for every fixed $z \in Z_\infty$, when t ranges over $\tilde{T}_\infty$.

In addition we require that V is invariant under multiplication on the left by the subgroup $U_0 = U \cap Z_A$, that is,

$$U_0 V = V.$$

Here U is the subgroup of adeles of the form $(1, u_2, \dots u_p, \dots)$, $u_p \in U_p$.

We show that under this condition the image of a Siegel set S in X contains together with every point x at least one horosphere passing through it.

Let x be a point of X belonging to the image of a Siegel set S, and $g = ztv$, $z \in Z_\infty$, $t \in \tilde{T}_\infty$, $v \in V$ be one of its inverse images in S. We consider the set $Z_\infty t U_0 v$. This set is contained in S, and its image in X is a horosphere. This follows from the fact that the projection of $Z_\infty U_0$ onto $Z_Q \setminus Z_A$ fills the whole of $Z_Q \setminus Z_A$.

In this subsection we show, on the basis of two results of Borel, that *there exists a Siegel set S whose image under the natural map onto $X = G_Q \setminus G_A$ coincides with the whole of X.* In other words, we show that the following decomposition holds:

$$G_A = G_Q Z_\infty \tilde{T}_\infty V, \tag{4}$$

where V is a compact set and $\tilde{T}_\infty$ a semibounded subset of T_∞.

The decomposition (4) is a consequence of the following results of Borel. In [4] the following decomposition is established

$$G_\infty = \bigcup_{i=1}^{n} G_Z \gamma_i Z_\infty \tilde{T}_\infty^0 V_\infty, \tag{5}$$

where V_∞ is a compact set in G_∞. Furthermore, as shown in [5], there exists a finite set of elements $x_1, \ldots, x_m \in G_A$ such that

$$G_A = \bigcup_{k=1}^{m} G_Q x_k G_A^\infty, \tag{6}$$

where G_A^∞ denotes the subgroup of G_A consisting of the elements of G_A of the form

$$(g_\infty, u_2, u_3, \ldots), \ g_\infty \in G_\infty, \ u_p \in U_p.$$

Here U_p denotes the integral subgroup of G_p.

It is not hard to see that the groups G_A^∞ and $g^{-1}G_A^\infty g$, where $g \in G_Q$, are commensurable, that is, their intersection is of finite index in each of them.

Consequently, for every $x \in G_A$ there exists a finite set of elements $x_1, \ldots, x_m$ in G_A such that

$$xG_A^\infty \subset \bigcup_{k=1}^{m} G_A^\infty x_k.$$

Hence, there exists a finite set of elements $y_1, \ldots, y_m$ in G_A such that

$$G_A = \bigcup_{k=1}^{m} G_Q G_A^\infty y_k. \tag{7}$$

The decomposition (4) follows immediately from (5) and (7).

In what follows we also use Siegel sets connected with Π-horospheres. They are defined as follows. Let Z^Π be a horospherical subgroup of G, and N^Π its normalizer. As we have shown in **§ 7.1**, $N^\Pi = G^\Pi Z^\Pi$, where G^Π is a reductive group. We denote by T^Π a maximal torus that lies in the center of G^Π and splits over Q. Just as this was done for Z, we introduce the groups Z_∞^Π and T_∞^Π.

Sets of the form

$$Z_\infty^\Pi \tilde{T}_\infty^\Pi V, \tag{8}$$

where $\tilde{T}_\infty^\Pi$ is a semibounded set in T_∞^Π and V a compact set in G_A, are called Siegel sets corresponding to the Π-horosphere.

Here the set V is always subject to the additional condition:

$$U^\Pi V = V,$$

where $U^\Pi = Z_A^\Pi \cap U$.

Under these conditions, as in the case of Siegel sets connected with maximal horospherical subgroups Z_A, the following proposition holds.

The image of a Siegel set contains together with every point x at least one Π*-horosphere passing through it.*

4. Regular Siegel Sets. Let

$$S = Z_\infty \tilde{T}_\infty V$$

be a Siegel set. We recall that V is assumed to be invariant under multiplication on the left by the group $U_0 = U \cap Z_A$, where U is the subgroup of adeles of the form $(1, u_2, \ldots, u_p, \ldots)$, $u_p \in U_p$. As we mentioned in **§ 7.3**, under this assumption the image of S in X contains together with every point an entire horosphere passing through it.

Note that these horospheres, in general, intersect each other.

It is not hard to verify that for the image of a Siegel set S in X to split into pairwise disjoint horospheres, the following condition is sufficient:

There exists a neighborhood W of the unit element of G_A such that if $g_1^{-1}\gamma g_2 \in W$, where $\gamma \in G_Q$, $g_1, g_2 \in S_1$ then $\gamma \in \Delta$, where Δ is the set of integral matrices in Z_Q.

We know S splits into sets of the form $Z_\infty U_0 t_1 v_1$. As we mentioned previously, the projections of these sets are horospheres. We show that the horospheres corresponding to distinct values of $t_1 v_1$ modulo U_0 do not intersect. Suppose that this is not so, that is, the projections of the sets $Z_\infty U_0 t_1 v_1$ and $Z_\infty U_0 t_2 v_2$ intersect. This means that there exists a $\gamma \in G_Q$ such that

$$\gamma z_1 u_1 t_1 v_1 = z_2 u_2 t_2 v_2.$$

From the stated condition it follows that $\gamma \in \Delta \subset Z_\infty U_0$; hence, $t_1 v_1 = u t_2 v_2$ where $u \in U_0$.

Siegel sets satisfying the condition are called *regular*. The aim of this subsection is to prove the following proposition.

Consider a Siegel set of the following form:

$$S = Z_\infty \tilde{T}_\infty U_0 g W_0, \tag{1}$$

where g is fixed element of G_A, W_0 a compact neighborhood of the unit element, and $\tilde{T}_\infty$ a semibounded set in T_∞.

If the neighborhood W_0 is sufficiently small and the elements t of $\tilde{T}_\infty$ satisfy the condition:

$$\alpha(\ln t) > c \text{ for every simple root } \alpha,$$

where c is a sufficiently large number, then S is a regular Siegel set.

Let

$$g_1 = z_1 t_1 u_1 g w_1, \quad g_2 = z_2 t_2 u_2 \delta w_2 \tag{2}$$

be two elements in S (here $z_1, z_2 \in Z_\infty$, $t_1, t_2 \in \tilde{T}_\infty$, $u_1, u_2 \in U_0$, w_1,

$w_2 \in W_0$), and let $\gamma \in G_Q$. We have to show that the condition

$$g_1^{-1}\gamma g_2 \in W, \tag{3}$$

that is,

$$w_1^{-1}g^{-1}u_1^{-1}t_1^{-1}z_1^{-1}\gamma z_2 t_2 u_2 g w_2 \in W, \tag{4}$$

where W is a fixed neighborhood of the unit element G_A (this neighborhood will be determined later), implies that $\gamma \in \Delta$.

We denote by F_∞ a fundamental domain in Z_∞ relative to the subgroup of integral matrices in Z_∞. We know that this domain is compact.

We represent z_1 and z_2 in the form

$$z_1 = \delta_1 z_1', \quad z_2 = \delta_2 z_2',$$

where

$$\delta_1 = (\delta_1, 1, \ldots, 1, \ldots), \quad \delta_2 = (\delta_2, 1, \ldots, 1, \ldots),$$

δ_1 and δ_2 are integral matrices, and z_1', $z_2' \in F$.

Then condition (4) can be rewritten in the form

$$w_1^{-1}g^{-1}u_1^{-1}t_1^{-1}z_1'^{-1}(\delta_1^{-1}\gamma\delta_2) z_2' t_2 u_2 g w_2 \in W. \tag{5}$$

From (5) it follows that

$$t_1^{-1}\delta_1^{-1}\gamma\delta_2 t_2 \in (t_1^{-1}z_1't_1) U_0 g(W_0WW_0^{-1})g^{-1}U_0(t_2^{-1}z_2't_2). \tag{6}$$

We consider the projection of the element $t_1^{-1}\delta_1^{-1}\gamma\delta_2 t_2$ onto the subgroup G_a of adeles of the form $(1, g_2, \ldots, g_p, \ldots)$. Since the projections of the elements t_1, t_2, δ_1, δ_2, z_1, z_2 onto G_a are equal to 1, we obtain from (6) that

$$\gamma_a \in U_0 g_a(W_0WW_0^{-1})_a g_a^{-1}U_a. \tag{7}$$

(The subscript a signifies projection onto G_a.)

We assume the neighborhoods W_0 and W to be chosen so small that

$$g_a(W_0WW_0^{-1})g_a^{-1} \subset U. \tag{8}$$

Then it follows from (7) that

$$\gamma_a \in U,$$

where $\gamma_a = (1, \gamma, \ldots, \gamma, \ldots)$, and so γ is an integral matrix.

Now we consider the projection of the element $t_1^{-1}\delta_1^{-1}\gamma\delta_2 t_2$ onto G_∞.

From (6) we find that

$$t_1^{-1}\delta_1^{-1}\delta_2 t_2 \in (t_1^{-1}z_1't_1)g_\infty(W_0WW_0^{-1})_\infty g_\infty^{-1}(t_2^{-1}z_2't_2). \tag{9}$$

We recall that the elements z_1' and z_2' lie in a compact set. Since $\tilde{T}_\infty$ is semibounded, it follows that $t_1^{-1}z_1't_1$ and $t_2^{-1}z_2't_2$ lie in a sufficiently small neighborhood of the unit element. Thus, (9)

means that

$$t_1^{-1}\delta_1^{-1}\gamma\delta_2 t_2 \in K,$$

where K is a sufficiently small neighborhood of the unit element and γ', as we have already shown, an integral matrix.

The assertion of the theorem now follows immediately from the following lemma.

LEMMA. *If the elements t_1 and t_2 satisfy for every positive root α the inequality*

$$\alpha(\ln t) > c,$$

where c is a sufficiently large number, and if K is a sufficiently small neighborhood of the unit element, then the condition

$$t_1^{-1}\gamma t_2 \in K,$$

where γ is an integral matrix, implies that $\gamma \in Z$.

Proof. We consider the adjoint representation of G_∞. We also assume that the basis in the Lie algebra $\mathfrak{G}$ of G_∞ is consistent with its decomposition into root spaces:

$$\mathfrak{G} = \mathfrak{G}_0 + \sum_\alpha \mathfrak{G}_\alpha.$$

We may assume that precisely this matrix representation of G_∞ is the one taken initially.† Thus, γ is an element from the subgroup of integral matrices in the adjoint representation.

In accordance with the decomposition $\mathfrak{G} = \mathfrak{G}_0 + \sum \mathfrak{G}_\alpha$ we write the matrices $g \in G$ in block-diagonal form: $g = \|\gamma_{\alpha\beta}\|$.

We have to show that $\gamma_{\alpha\beta} = 0$ when $\alpha < 0$, $\beta > 0$ and that $\gamma_{\alpha\alpha}$ is the unit matrix.

Observe that in the chosen basis the matrices $t \in T$ are block-diagonal, and their diagonal elements have the following form:

$$t_{\alpha\alpha} = \exp(\alpha(\ln t))e_\alpha,$$

where e_α is a unit matrix.

Thus, the elements of the matrix $\gamma' = t_1^{-1}\gamma t_2$ have the form

$$\gamma'_{\alpha\beta} = \exp[\beta(\ln t_2) - \alpha(\ln t_1)]\gamma_{\alpha\beta}. \tag{10}$$

Suppose then that $\alpha < 0$ and $\beta > 0$. By the condition of the lemma, for arbitrary corresponding elements $\tilde{\gamma}_{\alpha\beta}$ and $\tilde{\gamma}'_{\alpha\beta}$ the matrices $\gamma_{\alpha\beta}$ and $\gamma'_{\alpha\beta}$ satisfy the inequality

$$|\tilde{\gamma}'_{\alpha\beta}| > \exp(2c)\,|\tilde{\gamma}_{\alpha\beta}|. \tag{11}$$

Assume that $\gamma_{\alpha\beta} \neq 0$. Since the nonzero elements of $\gamma_{\alpha\beta}$ are integers, they are bounded below in modulus. But then, by (11),

† This proposition is justified by the well-known fact that the subgroups of integral matrices in various representations of G are commensurable.

the elements of $\gamma'_{\alpha\beta}$ cannot be arbitrarily small, and this contradicts the condition of the lemma. So we have shown that $\gamma_{\alpha\beta} = 0$ when $\alpha < 0$, $\beta > 0$. Now we show that $\gamma_{\alpha\alpha}$ is a unit matrix. Observe that since γ is a triangular matrix, as we have already shown, the fact that γ and γ^{-1} are integral implies that $\gamma_{\alpha\alpha}$ is a unimodular matrix. Condition (10) gives us:

$$\gamma'_{\alpha\alpha} = \exp\left(\alpha\left(\ln \frac{t_2}{t_1}\right)\right)\gamma_{\alpha\alpha}.$$

Hence, it follows that

$$\det \gamma'_{\alpha\alpha} = \exp^{n_\alpha}\left(\alpha\left(\ln \frac{t_2}{t_1}\right)\right) \det \gamma_{\alpha\alpha} = \exp^{n_\alpha}\left(\alpha\left(\ln \frac{t_2}{t_1}\right)\right), \quad (12)$$

where n_α is the order of $\gamma_{\alpha\alpha}$.

Since the matrices $\gamma'_{\alpha\alpha}$ are by assumption sufficiently near to the unit matrix, the value of $\exp(\alpha(\ln(t_2 t_1^{-1})))$ is, by (12), sufficiently near to 1.

Consequently, the matrix

$$\gamma_{\alpha\alpha} = \exp\left(-\alpha\left(\ln \frac{t_2}{t_1}\right)\right)\gamma'_{\alpha\alpha}$$

is sufficiently near to the unit matrix. But since, furthermore, $\gamma_{\alpha\alpha}$ is an integral matrix, it must coincide with unit matrix, and the lemma is proved.

5. Regular Siegel Sets Connected with Π-Horospheres. Let

$$S = Z_\infty^\Pi \tilde{T}_\infty^\Pi V$$

be a Siegel set connected with the Π-horospheres. We recall that by assumption the compact set V satisfies the following condition

$$U^\Pi V = V,$$

where $U^\Pi = U \cap Z_A$, U is the subgroup of adeles of the form $(1, u_2, \ldots, u_p, \ldots)$, $u_p \in U_p$. Under this assumption the image of S in X contains with every point x the entire Π-horosphere passing through it. These horospheres intersect, in general.

It is not hard to verify that the following condition is sufficient for the image of the Siegel set S in X to split into pairwise disjoint Π-horospheres:

There exists a neighborhood W of the unit element in G_A such that if $g_1^{-1}\gamma g_2 \in W$, where $g_1, g_2 \in W$, $\gamma \in G_Q$, then $\gamma \in \Delta^\Pi$, where Δ^Π denotes the integral subgroup of Z_Q^Π.

The proof of this proposition proceeds just as for Siegel sets connected with maximal horospherical subgroups Z_A (see **§ 7.4**).

Siegel sets satisfying this condition are called *regular*. The following result is established following the pattern of **§ 7.4** for the case of maximal horospherical subgroups:

Consider a Siegel set of the following form

$$S = Z_\infty^\Pi \tilde{T}_\infty^\Pi U^\Pi g W_0,$$

where g is a fixed element in G_A, W_0 a compact neighborhood of the unit element, and $\tilde{T}_\infty^\Pi$ a semibounded set in T_∞^Π.

If the neighborhood W_0 is sufficiently small and the elements t of the set $\tilde{T}_\infty^\Pi$ satisfy the condition:

$$\alpha(\ln t) > c$$

for every simple root α, where c is a sufficiently large number, then S is a regular Siegel set.

In this subsection we establish the following result:

THEOREM. *The space $X = G_Q \setminus G_A$ can be covered by the projections of a finite number of regular Siegel sets.*

Proof. As shown in **§ 7.3**, there exists a Siegel set

$$S = Z_\infty \tilde{T}_\infty V,$$

whose projection onto $X = G_Q \setminus G_A$ coincides with X. Let Π be an arbitrary subset of the set of simple roots. We denote by $\tilde{T}_\infty(\Pi)$ the subset of $\tilde{T}_\infty$ consisting of those t for which

$$\alpha_i(\ln t) > c, \qquad \text{when } \alpha_i \in \Pi;$$

$$\alpha_i(\ln t) < c, \qquad \text{when } \alpha_i \notin \Pi,$$

where c is sufficiently large.

Then it is obvious that

$$S = \sum_\Pi Z_\infty T_\infty(\Pi) V.$$

We show that for each of the subsets $Z_\infty \tilde{T}_\infty(\Pi) V$ there exists a Siegel set

$$S^\Pi = Z_\infty^\Pi \tilde{T}_\infty^\Pi V^\Pi,$$

whose projection onto X contains the projection of the set $Z_\infty \tilde{T}_\infty(\Pi) V$, and such that

$$\alpha(\ln t) > c$$

for every simple root α and every $t \in \tilde{T}_\infty$, where c is the constant determined above.

Observe that the elements g and δg, where $\delta \in \Delta$, have one and the same projection onto X. Therefore, the projection onto X of the set $Z_\infty \tilde{T}_\infty(\Pi) V$ coincides with the projection of $F_\infty \tilde{T}_\infty(\Pi) V$, where F_∞ is a fundamental domain of Z_∞ relative to the subgroup Δ of integral matrices. This domain F_∞ is a compact set.

We split Z into the semidirect product

$$Z_\infty = Z_\infty^\Pi \tilde{Z}_\infty^\Pi$$

of the subgroup Z_∞^Π and a complementary subgroup $\tilde{Z}_\infty^\Pi$.† We denote by $\tilde{F}_\infty^\Pi$ the projection of F_∞ onto $\tilde{Z}_\infty^\Pi$. Obviously, $\tilde{F}_\infty^\Pi$ is a compact set; consequently,

$$\sum_{t \in \tilde{T}_\infty(\Pi)} t^{-1}\tilde{F}_\infty^\Pi t.$$

is also compact, because $\tilde{T}_\infty(\Pi)$ is semibounded.

On the other hand, it is easy to verify that the set $T_\infty(\Pi)$ can be represented in the form of a product

$$\tilde{T}_\infty(\Pi) = \tilde{T}_\infty^\Pi T',$$

where T' is a compact set and $\alpha(\ln t) > c$ for every simple root α and every $t \in T_\infty^\Pi$.

We set

$$V^\Pi = T'\Bigl(\sum_{t \in \tilde{T}_\infty(\Pi)} t^{-1}\tilde{F}_\infty^\Pi t\Bigr)V$$

and show that

$$S^\Pi = Z_\infty^\Pi \tilde{T}_\infty^\Pi V^\Pi$$

is the required set.

For let ztv be an element of $Z_\infty \tilde{T}(\Pi)V$. As we said earlier, we may assume that $z \in F_\infty$. Then we have $z = z_\infty^\Pi \tilde{z}_\infty^\Pi$, where $z_\infty^\Pi \in Z_\infty^\Pi$, $\tilde{z}_\infty^\Pi \in \tilde{F}_\infty^\Pi$. On the other hand, we have $t = t^\Pi t'$, where $t^\Pi \in \tilde{T}_\infty^\Pi$, $t' \in T'$.

Consequently,

$$ztv = z_\infty^\Pi t^\Pi t'(t^{-1}\tilde{z}_\infty^\Pi t)v.$$

From this decomposition it is clear that ztv belongs to S^Π, as required.

So we have shown that X is covered by the projections of a finite number of Siegel sets

$$Z_\infty^\Pi \tilde{T}_\infty^\Pi V,$$

such that $\alpha(\ln t) > c$ for every simple root α and every $t \in \tilde{T}_\infty^\Pi$.

To complete the proof of the theorem we remark that each of these sets is covered by finitely many sets of the form $Z_\infty^\Pi \tilde{T}_\infty^\Pi g_i W$, where W is a fixed sufficiently small neighborhood of the unit element. As was established earlier, if the constant c is taken sufficiently large and the neighborhood W sufficiently small, then $Z_\infty^\Pi \tilde{T}_\infty^\Pi g_i W$ are regular Siegel sets. Consequently, X can be covered by the projections of finitely many regular Siegel sets, as required.

6. Reduction of the Main Theorem. We now turn to the proof of the main theorem which asserts that for every positive definite

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† The Lie algebra of the group $\tilde{Z}^\Pi$ is generated by the root spaces $\mathfrak{G}$, where α ranges over the positive roots that are linear combinations of simple roots not occurring in Π.

Schwartz-Bruhat function $\varphi(g)$ the trace of the operator T_φ in H_π^0 is finite.

As we have shown in **§ 7.5**, there exist regular Siegel sets S_k whose projections X_k onto X cover the whole of X.

We consider the subspace $L_2(X_k)$ of functions $f(x) \in L_2(X)$ that are concentrated on X_k and denote by P_k the projection operator onto $L_2(X_k)$.

Obviously,

$$\operatorname*{Tr}_{H_\pi^0} T_\varphi \leq \sum_k \operatorname*{Tr}_{P_k H_\pi^0} (P_k T_\varphi P_k).$$

Consequently, for the proof of the main theorem it is sufficient to show that the trace of the operator $P_k T_\varphi P_k$ onto $P_k H_\pi^0$ is finite.

Now we make a further reduction of the main theorem.

Let

$$S_k = Z_\infty^\Pi \tilde{T}_\infty^\Pi V$$

be a regular Siegel set whose projection onto X is X_k.

We consider the collection $H_\pi^0(S_k)$ of all functions $f(g)$ that are concentrated on S_k and satisfy the following conditions:

1. $f(\delta g) = f(g)$ for every $\delta \in \Delta^\Pi$;
2. $f(kg) = \pi(k) f(g)$ for every $k \in K_A$;
3. $\displaystyle\int_{\Delta^\Pi \backslash S_k / K_A} |f(g)|^2 \, dg < \infty$;
4. $\displaystyle\int_{Z_Q^{\Pi_1} \backslash Z_A^{\Pi_1}} f(z^{\Pi_1} g) \, dz^{\Pi_1} = 0$ if $\Pi_1 \subset \Pi$.

We show that

$$P_k H_\pi^0 \subset H_\pi^0(S_k).$$

For by the map

$$S_k \to X_k,$$

every function $f(x) \in P_k H_\pi^0$ can be put in correspondence with a function $f_1(g)$ on S_k defined by the formula

$$f_1(g) = f(x_g),$$

where x_k is the projection of g into X_k. An immediate verification shows that $f_1(g) \in H_\pi^0(S_k)$ and that

$$f(x) \to f_1(g)$$

is an isometric map of $P_k H_\pi^0$ into $H_\pi^0(S_k)$. Consequently, this map can be regarded as an embedding.

By virtue of this embedding, the proof of the main theorem reduces to the proof of the following proposition.

The trace of T_φ in $H_\pi^0(S_k)$ is finite for every positive definite Schwartz-Bruhat function φ.

This proposition will be proved in the next subsections.

To simplify the exposition, the further arguments are given only for the case when G is semisimple. In this case K is trivial; hence, H^0_π is the whole space H^0.

The investigation for an arbitrary reductive group differs from that given below only by a more cumbersome notation.

We write the formula for the kernel of T_φ in the space $H(S_k) = L_2(\Delta \setminus S_k)$. This kernel has the following form:

$$K(g_1, g_2) = \sum_{\gamma \in G_Q} \varphi(g_1^{-1}\gamma g_2), \; g_1, g_2 \in S_k. \tag{1}$$

We assume that the function φ is concentrated in a sufficiently small neighborhood W of the unit element. Since S_k is a regular Siegel set, the condition that $g_1^{-1}\gamma g_2 \in W$, where $g_1, g_2 \in S_k$, $\gamma \in G_Q$, implies that $\gamma \in \Delta^\Pi$, where Δ^Π is the subgroup of integral matrices of Z_Q^Π. Thus, the summation in (1) is taken, in fact, over the elements $\gamma \in \Delta^\Pi$, that is

$$K(g_1, g_2) = \sum_{\delta \in \Delta^{\Pi}} \varphi(g_1^{-1}\,\delta g_2). \tag{2}$$

7. The p-Norm. The concept of the p-norm, which we now introduce, is important for the following investigation. Let $f(x) = f(x_1, \ldots, x_n)$ be a sufficiently often differentiable function of n real variables.

We use the term p-norm of $f(x)$, where $p = 0, 1, 2, \ldots$, for the following expression:

$$\|f(x)\|_p = \max_{i_1, \ldots, i_k} \left(\max_x \left| \frac{\partial^{pk} f(x)}{\partial x_{i_1}^p \cdots \partial x_{i_k}^p} \right|\right), \tag{1}$$

the maximum being taken over all points x and all subsets $(i_1, \ldots i_k)$ of the index set $1, \ldots n$.

Note that the concept of p-norm we have introduced is not invariant relative to the coordinate system. Therefore, in introducing the p-norm we must fix a definite system of coordinates.

We need the concept of p-norm to estimate the sum of the moduli of Fourier coefficients of $f(x)$. For let us assume that $f(x)$ is a periodic function with period 1 in each of the variables x_k and that c_m, $m = (m_1, \ldots, m_n)$ are its Fourier coefficients:

$$c_m = \int_0^1 \cdots \int_0^1 f(x) e^{2\pi i(m_1x_1 + \cdots + m_nx_n)}\, dx_1 \cdots dx_n. \tag{2}$$

We show that then the following estimate holds:

$$\sum_{m \neq 0} |c_m| \leq C^{(p)} \|f\|_p, \tag{3}$$

where $C^{(p)}$ is a constant, $p = 2, 3, \ldots$.

For consider the Fourier coefficients $c_{m'}$, $m' = (m_1, \ldots, m_n)$ in which $m_{i_1}, \ldots, m_{i_k}$ are different from zero and $m_i = 0$ for $i \neq i_1, \ldots, i_k (i_1, \ldots, i_k$ is a fixed subset of indices). Integrating by parts, we find the following estimate for these coefficients $c_{m'}$:

$$|c_{m'}| \leq C(m_{i_1} \cdots m_{i_k})^{-p} \max \left| \frac{\partial^{pk} f(x)}{\partial x_{i_1}^p \cdots \partial x_{i_k}^p} \right|.$$

Hence, summing over all nonzero $m_{i_1}, \ldots, m_{i_k}$, we obtain

$$\sum_{m'} |c_{m'}| \leq C_1 \max_x \left| \frac{\partial^{pk} f(x)}{\partial x_{i_1}^p \cdots \partial x_{i_k}^p} \right|. \tag{4}$$

Finally, summing the inequality (4) over all subsets of indices $(i_1, \ldots i_k)$, we obtain the required estimate (3).

8. Proof of the Main Theorem. The operator T_φ is considered in the space $H = L_2(\Delta^\Pi \setminus S_k)$, where $S_k = Z_\infty^\Pi \tilde{T}_\infty^\Pi V$ is a regular Siegel set and φ a positive definite function on G_A that is concentrated in a sufficiently small neighborhood of the unit element. The kernel of this operator has the following form:

$$K(g_1, g_2) = \sum_{\delta \in \Delta \Pi} \varphi(g_1^{-1} \delta g_2). \tag{1}$$

In the preceding subsections we have reduced the proof of the main theorem to proving that the trace of T_φ on the subspace $H^0(\Delta \setminus S_k)$ is finite. This proof begins here and is completed in **§ 7.10.**

In the Lie algebra $\mathfrak{Z}_\infty^\Pi$ of the group Z_∞^Π we choose a system of coordinates that is defined over Q and compatible with the decomposition of $\mathfrak{Z}_\infty^\Pi$ into root spaces. Making use of the canonical map

$$t \to \exp t$$

of $\mathfrak{Z}_\infty^\Pi$ onto Z_∞^Π we transfer this system of coordinates to Z_∞^Π.

We denote by $k_p(g)$ the p-norm of the function

$$\varphi(z; g) = K(zg, g),$$

regarded as a function on Z_∞^Π. We now show that there exists a p such that

$$\int_{\Delta^\Pi \setminus S_k} k_p(g)\, dg < \infty. \tag{2}$$

First of all we estimate the number of nonzero terms in the series for $K(zg, g)$:

$$K(zg, g) = \sum_{\delta \in \Delta \Pi} \varphi(g^{-1} z^{-1} \delta g), \quad g = z'tv. \tag{3}$$

By assumption for such terms we must have

$$v^{-1}t^{-1}z'^{-1}\,\delta z'tv \in W,$$

where W is a sufficiently small neighborhood of the identity.

We rewrite this condition in the following form:

$$v^{-1}(t^{-1}z't)^{-1}(t^{-1}z^{-1}\,\delta t)(t^{-1}z't)v \in W. \tag{4}$$

Since z' belongs to a bounded set and t ranges over a semibounded set on a torus, $t^{-1}z't$ belongs to a bounded set. Since v also belongs to a bounded set, by condition (4) we have

$$t^{-1}z^{-1}\,\delta t \in W',$$

hence,

$$t^{-1}\,\delta t \in (t^{-1}zt)W', \tag{5}$$

where W' is a sufficiently small neighborhood of the identity. We denote by $\varphi(z)$ the maximum of the moduli of the matrices z outside the main diagonal. Then we obtain from (5) the following estimate:

$$\varphi(t^{-1}\,\delta t) < C\varphi(z). \tag{6}$$

(We use the fact that, since the set of elements t is semibounded, we have $\varphi(t^{-1}zt) < C_1\varphi(z)$.)

Observe that under the transformation $\delta \to t^{-1}\,\delta t$ every nonzero element outside the main diagonal of δ is multiplied by $t^{-\alpha} = \exp(-\alpha(\ln t))$, where α is a positive root (the same for every element of δ). Consequently, on the basis (6) we obtain the following estimate:

$$\varphi(\delta) < C\varphi(z)t^{\alpha_0}, \tag{7}$$

where $t^{\alpha_0} = \exp(\alpha_0(\ln t))$, and α_0 is the sum of all positive roots.

Obviously, the number N of integral matrices δ satisfying (7) is subject to the following estimate:

$$N < C\varphi(z)t^{n\alpha_0},$$

where n is the dimension of the group Z^{Π}.

So we have established that the number of nonzero terms of the series (3) does not exceed $C\varphi(z)t^{n\alpha_0}$, where $\alpha_0 = \sum_{\alpha>0}\alpha$, and n is the dimension of Z^{Π}.

Now it is very easy to prove the convergence of the series (3) for sufficiently large p. To see this we observe that the p-th derivative of every term in (3) does not exceed in modulus the number

$$\min_{\alpha>0} t^{-p\alpha} \le t^{-(p/a)\alpha_0},$$

where $\alpha_0 = \sum_{\alpha>0}\alpha$, and a is the total number of roots. Consequently,

the p-th derivative of the sum of (3) does not exceed

$$C\varphi(z)t^{[n-(p/a)]\alpha_0}.$$

So we have established that

$$k_p(g) \leq C\varphi(z)t^{[n-(p/a)]\alpha_0}.$$

By virtue of this estimate it is obvious that the integral

$$\int\limits_{\Delta^{\text{II}}\backslash s_k} k_p(g)\, dg = \int\limits_{\Delta^{\text{II}}\backslash Z^{\Pi}} \int\limits_{T} \int\limits_{V} k_p(ztv)\, dz\, dt\, dv$$

converges if p is sufficiently large.

We conclude the proof of the main theorem in **§ 7.10.** In **§ 7.9** we state, and in **§ 7.10** we prove, a fundamental lemma from which the finiteness of the trace of the operator satisfying condition (2) follows immediately. This lemma refers to a certain class of integral operators on soluble groups.

9. Solvable Algebras and Groups. Statement of the Fundamental Lemma. Let $\mathfrak{R}$ be an algebraic Lie algebra, solvable and splitting over the field Q of rational numbers. As is well known, $\mathfrak{R}$ admits the Chevalley decomposition

$$\mathfrak{R} = \mathfrak{Z} + \mathfrak{T},$$

where $\mathfrak{Z}$ is the maximal nilpotent ideal and $\mathfrak{T}$ a commutative subalgebra; thus,

$$[\mathfrak{Z}, \mathfrak{T}] \subset \mathfrak{Z} \quad \text{and} \quad [\mathfrak{T}, \mathfrak{T}] = 0.$$

In the nilpotent subalgebra $\mathfrak{Z}$ we can introduce in a natural way the concept of a root and of a root space. For we consider linear functions $\alpha(t)$ on $\mathfrak{T}$ with values in Q. We call a function $\alpha(t)$ a *root* if $\mathfrak{Z}$ contains a vector $\mathfrak{z} \neq 0$ such that

$$[\mathfrak{z}, t] = \alpha(t)\,\mathfrak{z} \text{ for every } t \in \mathfrak{T}.$$

Obviously, the totality of vectors $\mathfrak{z}$ corresponding to one and the same root $\alpha(t)$ forms a linear subspace of $\mathfrak{Z}$. We call it a *root subspace* and denote it by $\mathfrak{Z}_\alpha$.

From the definition it follows that

$$[\mathfrak{Z}_\alpha, \mathfrak{Z}]_\beta \subset \mathfrak{Z}_{\alpha+\beta}$$

for arbitrary roots α and β. In particular, if $\alpha + \beta$ is not a root, then $[\mathfrak{Z}_\alpha, \mathfrak{Z}_\beta] = 0$.

We call a root α of the algebra $\mathfrak{Z}$ simple if it cannot be represented as a sum of other roots.

We assume that the set of roots α of $\mathfrak{Z}$ satisfies the following conditions:

1. The simple roots form a linearly independent system Π_0.
2. Every root of $\mathfrak{Z}$ is representable as the sum of simple roots.

An algebra $\mathfrak{Z}$ satisfying these conditions is called *regular.*

Furthermore, we always assume that the number of simple roots of $\mathfrak{Z}$ is equal to the dimension of the space $\mathfrak{T}$.

We proceed from the algebra $\mathfrak{R} = \mathfrak{Z} + \mathfrak{T}$ over Q, to its corresponding solvable group over the field of real numbers:

$$R = ZT,$$

where Z is the unipotent normal subgroup of R corresponding to $\mathfrak{Z}$, and T the torus corresponding to the commutative subalgebra $\mathfrak{T}$.

We denote by Δ a discrete subgroup of Z that is of finite index in the group of all integral matrices in Z.

We recall the definition of a semibounded set on the torus T. As set $T^0 \subset T$ is called semibounded if for every fixed $z \in Z$ the set of elements of the form

$$t^{-1}zt, \quad t \in T^0$$

is bounded.

Let

$$X = \Delta \setminus ZT^0,$$

where T^0 is a semibounded set on the torus, and Δ a subgroup of finite index of the group of integral matrices in Z.

We consider the space $L_2(X)$, that is, the space of functions $f(z, t)$, $z \in Z$, $t \in T^0$, satisfying the following conditions:

1. $f(\delta z, t) = f(z, t)$ for every $\delta \in \Delta$;
2. $\displaystyle\int_{\Delta\setminus ZT^0} |f(z, t)|^2\, dz\, dt < \infty.$

Thus, the space $L_2(X)$ is a tensor product

$$L_2(X) = L_2(\Delta \setminus Z) \otimes L_2(T^0).$$

In $L_2(\Delta \setminus Z)$ we select the subspace H_0 of functions whose integrals over every horosphere on $\Delta \setminus Z$ vanish, and we consider the subspace

$$\tilde{H}_0 = H_0 \otimes L_2(T^0)$$

of $L_2(X)$ corresponding to H_0.

Our next task is the proof of the lemma on the finiteness of the trace of the operator on $\tilde{H}_0$ that is stated below.

We call an integral operator A on $L_2(X)$ with the kernel $K(z_1t, z_2t)$ *regular* if the following conditions are satisfied:†

† It is implicit that $K(\delta_1 z_1 t_1, \delta_2 z_2 t_2) = K(z_1 t_1, z_2 t_2)$ for arbitrary $\delta_1, \delta_2 \in \Delta$.

1. A is a self-adjoint positive operator.

2. The kernel K is an infinitely differentiable function on $X \otimes X$.

3. We set

$$k_p(z^0, t) = \|f(z)\|_p, \tag{1}$$

where

$$f(z) = K(zt, z^0 t),$$

and $\|f\|_p$ is the p-norm of the function $f(z) \equiv f(\tau_1, \ldots, \tau_n)$ defined in **§ 7.7**. Then we can find a p such that

$$\int\limits_{\Delta \backslash ZT^0} k_p(z, t)\, dz\, dt < \infty.$$

The following proposition holds.

Lemma. *The trace of a regular operator A on the subspace $\tilde{H}_0$ is finite.*

By the trace of the operator A on a subspace we mean the trace of the operator PAP, where P is the projection operator onto the given space.

10. Proof of the Fundamental Lemma. The proof proceeds by induction on the number of roots of Z.

We consider an arbitrary root space $\mathfrak{Z}_m$ belonging to the center of the algebra $\mathfrak{Z}: [\mathfrak{Z}, \mathfrak{Z}_m] = 0$. Clearly, $\mathfrak{Z}_m$ is a subalgebra of $\mathfrak{Z}$. To this subalgebra $\mathfrak{Z}_m$ there corresponds a subgroup Z_m belonging to the center of Z.

Let χ range over the set of characters of the compact commutative group $\Delta_m \setminus Z_m$; H^χ the subspace of functions $f(z)$ in $H = L_2(\Delta \setminus Z)$ for which

$$f(z_m z) = \chi(z_m) f(z) \text{ for every } z_m \in Z_m.$$

Obviously, the space $H = L_2(\Delta \setminus Z)$ is the direct sum of the subspaces H^χ:

$$H = \sum_\chi H^\chi.$$

In H we single out the subspace

$$H' = \sum_{\chi \neq \chi_0} H^\chi,$$

where χ_0 is the character identically equal to 1. We show that the trace of the operator A on the subspace $\tilde{H}' = H' \otimes L_2(T^0)$ is finite.

For let P_χ be the projection operator onto $H^\chi \otimes L_2(T^0)$:

$$P_\chi f(z, t) = \int\limits_{\Delta_m \backslash Z_m} \bar{\chi}(z_m) f(z_m z, t)\, dz_m.$$

Then we have

$$\operatorname*{Tr}_{\tilde{H}\chi} A = \operatorname{Tr}(P_\chi A P_\chi) = \int_X \int_{\Delta_m \setminus Z_m} \bar{\chi}(z_m) K(z_m zt, zt)\, dz_m\, dz_m\, dz\, dt.$$

Since

$$\operatorname*{Tr}_{\tilde{H}'} A \le \sum_{\chi \neq \chi_0} \operatorname*{Tr}_{\tilde{H}\chi} A,$$

thus,

$$\operatorname*{Tr}_{\tilde{H}'} A \le \sum_{\chi \neq \chi_0} \int_X \int_{\Delta_m \setminus Z_m} \bar{\chi}(z_m) K(z_m zt, zt)\, dz_m\, dz\, dt. \tag{1}$$

From (1) it follows that

$$\operatorname*{Tr}_{\tilde{H}'} A \le \int_X a(z, t)\, dz\, dt,$$

where

$$a(z, t) = \sum_{\chi \neq \chi_0} \left| \int_{\Delta_m \setminus Z_m} \bar{\chi}(z_m) K(z_m zt, zt)\, dz_m \right|.$$

Let us estimate $a(z, t)$. Let $\|K(z_m zt, zt\|_p$ be the p-norm of the function $K(z_m zt, zt)$, regarded as a function of z_m. On the basis of the estimate for the Fourier coefficients derived in **§ 7.7** we then have

$$a(z, t) \le C\, \|K(z_m zt, zt)\|_p.$$

But, as is easy to verify,

$$\|K(z_m zt, zt)\|_p \le k_p(z, t),$$

where $k_p(z, t)$ is the function defined by formula (1) in **§ 7.9**. So we have

$$a(z, t) \le C k_p(z, t),$$

and therefore

$$\operatorname*{Tr}_{\tilde{H}'} A \le \int_X k_p(z, t)\, dz\, dt$$

for every p. By the assumption of the lemma there exists a p for which $\int_X k_p(z, t)\, dz\, dt < \infty$. Consequently, $\operatorname*{Tr}_{\tilde{H}'} A$ is also finite.

Now we proceed directly to the proof of the lemma: that the trace of A on the subspace $\tilde{H}_0 = H_0 \otimes L_2(T^0)$ is finite.

We decompose $\tilde{H}_0$ into the direct sum of subspaces

$$\tilde{H}_0^\chi = H_0^\chi \otimes L_2(T^0),$$

where $H_0^\chi = H^\chi \cap H_0$.

Since

$$\tilde{H}_0' \doteq \sum_{\chi \neq \chi_0} \tilde{H}_0^\chi \subset \tilde{H}',$$

we have

$$\operatorname*{Tr}_{\tilde{H}_0'} A \le \operatorname*{Tr}_{\tilde{H}'} A.$$

Consequently, by what was proved previously, A has a finite trace on the subspace $\tilde{H}_0' = \sum_{\chi \neq \chi_0} \tilde{H}_0^{\chi}$.

Thus, to complete the proof of the lemma it suffices to verify that A also has a finite trace on the subspace

$$\tilde{H}_0^{\chi_0} = H_0^{\chi_0} \otimes L_2(T^0).$$

Let us show this.

Observe that since $\tilde{H}_0^{\chi_0} \subset \tilde{H}^{\chi_0}$,

$$\operatorname*{Tr}_{\tilde{H}_0^{\chi_0}} A = \operatorname*{Tr}_{\tilde{H}_0^{\chi_0}} P_{\chi_0} A P_{\chi_0}$$

where P_{χ_0} is the projection operator onto $\tilde{H}^{\chi_0}$:

$$P_{\chi_0} f(z, t) = \int_{\Delta_m | Z_m} f(z_m z, t)\, dz_m.$$

Thus, instead of A we can consider the operator $P_{\chi_0} A P_{\chi_0}$ whose kernel is expressed in terms of the kernel K of A by the following formula:

$$K(z_1 t_1, z_2 t_2) = \int_{(\Delta_m \backslash Z_m) \times (\Delta_m \backslash Z_m)} K(z_m z_1 t_1, z_m' z_2 t_2)\, dz_m\, dz_m'.$$

Now let us find out how the spaces $\tilde{H}^{\chi_0} = H^{\chi_0} \otimes L_2(T^0)$ and $\tilde{H}_0^{\chi_0} = H_0^{\chi_0} \otimes L_2(T^0)$ are constructed.

The space H^{χ_0} consists of the functions $f(z) \in L_2(\Delta \setminus Z)$ that satisfy the condition

$$f(z_m z) = f(z) \text{ for every } z_m \in Z_m.$$

Consequently,

$$H^{\chi_0} = L_2(\Delta' \setminus Z'),$$

where $Z' = Z_m \setminus Z$, $\Delta' = Z_m \setminus Z_m \Delta \cong \Delta_m \setminus \Delta$.

In this realization $H_0^{\chi_0}$ is the space of all functions in $L_2(\Delta' \setminus Z')$ whose integrals over every horosphere on $\Delta' \setminus Z'$ vanish.

Observe that Z' is a regular group, for its simple roots are the same as those of Z. Since the total number of roots in Z' is one fewer than in Z (the last root has been deleted), we may, by the inductive hypothesis, assume the lemma to be proved for the space $\tilde{H}^{\chi_0}$.

Thus, if the regularity of the operator $P_{\chi_0} A P_{\chi_0}$ on $\tilde{H}^{\chi_0}$ is established, then it follows that its trace on the subspace $\tilde{H}_0^{\chi_0}$ is finite.

Obviously, the regularity conditions 1 and 2 hold for $P_{\chi_0} A P_{\chi_0}$. It remains to verify that condition 3 holds.

We consider the kernel K' of the operator $P_{\chi_0} A P_{\chi_0}$ defined by (2). We must interpret this kernel as a function on

$$(\Delta' \setminus Z' T^0) \otimes (\Delta' \setminus Z' T^0).$$

We set $f'(z) = K'(zt, z^0t)z$, $z^0 \in Z'$, $k'_p(z_0, t) = \|f'\|'_p$, where $\|f'(z)\|'_p$ denotes the p-norm of f regarded as a function of $z' \in Z'$. Then the regularity condition 3 for $P_{\chi_0}AP_{\chi_0}$ comes to

$$\int_{\Delta'\backslash Z'T^0} k'_p(z, t)\, dz\, dt < \infty$$

for a suitable p. It is not hard to verify that

$$k'_p(z, t) \leq \int_{\Delta_n\backslash Z_n} k_p(z_m z, t)\, dz_m,$$

where $k_p(z, t)$ is the function defined by (1) in **§ 7.9**. Consequently,

$$\int_{\Delta'\backslash Z'T^0} k'_p(z, t)\, dz\, dt \leq \int_{\Delta'\backslash Z'T^0} \int_{\Delta_m\backslash Z_m} k_p(z_m z, t)\, dz_m\, dz\, dt.$$

The integral on the right can be rewritten as

$$\int_{\Delta\backslash ZT^0} k_p(z, t)\, dz\, dt.$$

By assumption this integral is finite for some p. Consequently, for this p the integral

$$\int_{\Delta'\backslash Z'T^0} k'_p(z, t)\, dz\, dt.$$

is also finite. So we have proved the regularity of $P_{\chi_0}AP_{\chi_0}$ on $\tilde{H}^{\chi_0}$. By the inductive hypothesis, it follows that the trace of this operator on the subspace $\tilde{H}_0^{\chi_0}$ is finite, and the lemma is proved.

We now show that the main theorem follows from this lemma. For this purpose it is sufficient to verify that the integral operator discussed in **§ 7.8** is regular. Condition 1 obviously holds if the function φ is positive definite. Condition 2 holds if φ is a Schwartz-Bruhat function. The fact that condition 3 holds was verified in **§ 7.8.**

APPENDIX TO § 7

Functions on Regular Nilpotent Lie Groups

1. Regular Nilpotent Algebras. Let $\mathfrak{Z}$ be a nilpotent algebraic Lie algebra over the field Q of rational numbers. We call $\mathfrak{Z}$ a *graded algebra* if $\mathfrak{Z}$ splits into the direct sum of linear subspaces

$$\mathfrak{Z} = \sum_{\alpha \in \mathfrak{M}} \mathfrak{Z}_\alpha,$$

where α ranges over a finite subset $\mathfrak{M}$ of elements of a torsion-free abelian group. We also assume that the following conditions are satisfied:

$$[\mathfrak{Z}_\alpha, \mathfrak{Z}_\beta] \subset \mathfrak{Z}_{\alpha+\beta}, \qquad \text{when } \alpha + \xi \in \mathfrak{M},$$
$$[\mathfrak{Z}_\alpha, \mathfrak{Z}_\beta] = 0, \qquad \text{when } \alpha + \beta \text{ does not belong to } \mathfrak{M},$$

The subscripts $\alpha \in \mathfrak{M}$ are called *roots* of $\mathfrak{Z}$, and the corresponding subspaces $\mathfrak{Z}_\alpha$ the *root spaces* of $\mathfrak{Z}$.

We call a root α of $\mathfrak{Z}$ *simple* if it cannot be represented as the sum of other roots.

We say that a graded algebra $\mathfrak{Z}$ is *regular* if the following conditions hold:

1. Simple roots of $\mathfrak{Z}$ are linearly independent.
2. Every root of $\mathfrak{Z}$ is representable as a sum of simple roots.

Henceforth we consider only regular algebras $\mathfrak{Z}$.

Let

$$\Pi_0 : \alpha_1, \ldots, \alpha_n$$

be the system of simple roots of $\mathfrak{Z}$. By definition, every root α of $\mathfrak{Z}$ can be represented uniquely as a sum

$$\alpha = \sum c_i \alpha_i,$$

where the c_i are nonnegative integers.

With every pair of nonempty subsets Π', Π of the set Π_0 of all simple roots, $\Pi' \subset \Pi$, we associate a subalgebra $\mathfrak{Z}_\Pi^{\Pi'}$ of $\mathfrak{Z}$, which we define as follows:

Let $\mathfrak{M}_\Pi^{\Pi'}$ be the set of all roots of the form

$$\alpha = \sum_{\alpha_i \in \Pi} c_i \alpha_i, \tag{1}$$

where the sum is taken over the set of simple roots in Π and at least one root $\alpha_i \in \Pi'$ occurs in this sum with a nonzero coefficient. We set

$$\mathfrak{Z}_\Pi^{\Pi'} = \sum_{\alpha \in \mathfrak{M}_\Pi^{\Pi'}} \mathfrak{Z}_\alpha. \tag{2}$$

Obviously, $\mathfrak{Z}_\Pi^{\Pi'}$ is a subalgebra of $\mathfrak{Z}$. Observe that in our notation $\mathfrak{Z} = \mathfrak{Z}_{\Pi_0}^{\Pi_0}$. Furthermore, we set $\mathfrak{Z}_\Pi^0 = 0$, where the index 0 denotes the empty set. We call the $\mathfrak{Z}_{\Pi_0}^{\Pi}$ *horospherical subalgebras* of $\mathfrak{Z}$. It is not hard to see that horospherical subalgebras are ideals of $\mathfrak{Z}$. We mention that among the horospherical algebras of $\mathfrak{Z}$ there is $\mathfrak{Z}$ itself (because $\mathfrak{Z} = \mathfrak{Z}_{\Pi_0}^{\Pi_0}$) and the null subalgebra.

Note that $\mathfrak{Z}_\Pi^\Pi$ is a regular subalgebra of $\mathfrak{Z}$ whose system of simple roots is the set Π. The algebras $\mathfrak{Z}_\Pi^{\Pi'}$, $\Pi' \subset \Pi$, are also horospherical subalgebras of $\mathfrak{Z}_\Pi^\Pi$.

The verification of the following properties of the algebras $\mathfrak{Z}_\Pi^\Pi$ presents no difficulties:

1. If $\Pi_1 \subset \Pi_2$, $\Pi'_1 \subset \Pi'_2$, then

$$\mathfrak{Z}_{\Pi_1}^{\Pi'_1} \subset \mathfrak{Z}_{\Pi_2}^{\Pi'_2}. \tag{3}$$

2. If $\Pi'' \subset \Pi' \subset \Pi$, then there is a decomposition into a direct sum

$$\mathfrak{Z}_{\Pi}^{\Pi'} = \mathfrak{Z}_{\Pi}^{\Pi''} + \mathfrak{Z}_{\Pi-\Pi''}^{\Pi'-\Pi''}, \tag{4}$$

where $\Pi - \Pi''$ denotes the complement to Π'' in Π. In particular, every regular algebra $\mathfrak{Z}_{\Pi}^{\Pi}$ splits into the direct sum

$$\mathfrak{Z}_{\Pi}^{\Pi} = \mathfrak{Z}_{\Pi}^{\Pi'} + \mathfrak{Z}_{\Pi-\Pi'}^{\Pi-\Pi'}$$

of any horospherical subalgebra $\mathfrak{Z}_{\Pi}^{\Pi'}$ and the regular subalgebra $\mathfrak{Z}_{\Pi-\Pi'}^{\Pi-\Pi'} (\Pi' \subset \Pi)$.

2. Regular Nilpotent Lie Groups. Let $\mathfrak{Z}$ be an arbitrary nilpotent regular Lie algebra over Q, as defined in **§ 7.1,** Appendix 1, and Π_0 the set of its simple roots. We denote by Z the nilpotent algebraic group over the field of real numbers corresponding to $\mathfrak{Z}$. Accordingly, we denote by $Z_{\Pi}^{\Pi'}$, $\Pi' \subset \Pi \subseteq \Pi^0$, the algebraic subgroup over the field of real numbers corresponding to the subalgebra $\mathfrak{Z}_{\Pi}^{\Pi'}$. Let Δ be the subgroup of integral matrices of Z, or an arbitrary subgroup of finite index in the group of integral matrices. We set

$$\Delta_{\Pi}^{\Pi'} = \Delta \cap Z_{\Pi}^{\Pi'}.$$

In accordance with the terminology of **1** we call the subgroups $Z_{\Pi_0}^{\Pi}$ *horospherical subgroups* of Z, and the subgroups Z_{Π}^{Π} *regular* groups, among them the group $Z = Z_{\Pi_0}^{\Pi_0}$.

We list the basic properties of the groups $Z_{\Pi}^{\Pi'}$. First, from the results of **1** it follows immediately that:

1. If $\Pi_1 \subset \Pi_2$, $\Pi'_1 \subset \Pi'_2$, then

$$Z_{\Pi_1}^{\Pi'_1} \subset Z_{\Pi_2}^{\Pi'_2}, \quad \Delta_{\Pi_1}^{\Pi'_1} \subset \Delta_{\Pi_2}^{\Pi'_2} \tag{1}$$

2. If $\Pi'' \subset \Pi' \subset \Pi$, then we have decompositions into semi-direct products:

$$Z_{\Pi}^{\Pi'} = Z_{\Pi}^{\Pi''} Z_{\Pi-\Pi''}^{\Pi'-\Pi''}, \quad \Delta_{\Pi}^{\Pi'} = \Delta_{\Pi}^{\Pi''} \Delta_{\Pi-\Pi''}^{\Pi'-\Pi''}, \tag{2}$$

where $Z_{\Pi}^{\Pi''}$ is a normal subgroup of $Z_{\Pi}^{\Pi'}$.

Next, it is not difficult to check that *the spaces*

$$\Delta_{\Pi}^{\Pi'} \setminus Z_{\Pi}^{\Pi'}$$

are compact.

Since the groups $Z_{\Pi}^{\Pi'}$ are nilpotent, they possess an invariant measure $dz_{\Pi}^{\Pi'}$. We assume this measure to be normalized so that

$$\int_{\Delta_{\Pi}^{\Pi'} \setminus Z_{\Pi}^{\Pi'}} dz_{\Pi}^{\Pi'} = 1. \tag{3}$$

Our task is the investigation of the space $H = L_2(\Delta \backslash Z)$, that is, the space of functions $f(z)$ on Z satisfying the following conditions:

1. $$f(\delta z) = f(z) \qquad \text{for every} \qquad \delta \in \Delta; \tag{4}$$

2. $$\int_{\Delta\backslash Z} |f(z)|^2\, dz < \infty. \tag{5}$$

We now derive a decomposition of space H into a direct sum of subspaces that is important for representation theory (see the theorem below).

We introduce the concept of a horosphere in the space $\Delta \backslash Z$. Π-horospheres on Z are cosets $Z^{\Pi}_{\Pi_0} z$ of the horospherical subgroup $Z^{\Pi}_{\Pi_0}$ in Z. The images of Π-horospheres on Z under the natural map

$$Z \to \Delta \backslash Z$$

are called Π-horospheres in the space $\Delta \backslash Z$; they are obviously compact sets.

We now examine the following subspaces of H. Let H'_{Π} be the subspace of functions $f \in H$ that are constant on the Π-horospheres, that is, such that

$$f(z^{\Pi}_{\Pi_0} z) = f(z) \qquad \text{for every} \qquad z^{\Pi}_{\Pi_0} \in Z^{\Pi}_{\Pi_0}. \tag{6}$$

Next, let $H_{\Pi} \subset H'_{\Pi}$ be the subspace of functions that are constant on the Π-horospheres and such that their integrals over every Π'-horosphere with $\Pi' \supset \Pi$ vanish:

$$\int_{\Delta^{\Pi'}_{\Pi_0}\backslash z^{\Pi'}_{\Pi_0}} f(z^{\Pi'}_{\Pi} z)\, dz^{\Pi'}_{\Pi} = 0, \qquad \Pi' \supset \Pi. \tag{7}$$

Note that if $\Pi_1 \subset \Pi_2$, then $Z^{\Pi_1}_{\Pi_0} \subset Z^{\Pi_2}_{\Pi_0}$, and therefore

$$H'_{\Pi_1} \supset H'_{\Pi_2}.$$

There is a convenient way of describing the spaces H'_{Π} and H_{Π}. Since the group Z splits into the semidirect product

$$Z = Z^{\Pi}_{\Pi_0} Z^{\Pi_0-\Pi}_{\Pi_0-\Pi},$$

by condition (6) the functions $f \in H'_{\Pi}$ may be regarded as functions on $Z^{\Pi_0-\Pi}_{\Pi_0-\Pi}$. Thus,

$$H'_{\Pi} = L_2(\Delta^{\Pi_0-\Pi}_{\Pi_0-\Pi} \backslash Z^{\Pi_0-\Pi}_{\Pi_0-\Pi}). \tag{8}$$

Let us show that in this realization the subspace H_{Π} consists of all function f whose integrals over every horosphere in $\Delta^{\Pi_0-\Pi}_{\Pi_0-\Pi} \backslash Z^{\Pi_0-\Pi}_{\Pi_0-\Pi}$ vanish, that is,

$$\int_{\Delta^{\Pi'}_{\Pi_0-\Pi}\backslash z^{\Pi'}_{\Pi_0-\Pi}} f(z^{\Pi''}_{\Pi_0-\Pi} z)\, dz^{\Pi''}_{\Pi_0-\Pi} \equiv 0. \tag{9}$$

For suppose that $\Pi' \supset \Pi$, that is, $\Pi' = \Pi + \Pi''$, where $\Pi'' \subset \Pi_0 - \Pi$, $\Pi'' \neq 0$. Then we have the decomposition into the semidirect product

$$Z_{\Pi_0}^{\Pi'} = Z_{\Pi_0}^{\Pi} Z_{\Pi_0-\Pi}^{\Pi''}.$$

It is not hard to verify that for every function $f \in H$ the following integral relation holds:

$$\int\limits_{\Delta_{\Pi_0}^{\Pi'} \backslash Z_{\Pi_0}^{\Pi'}} f(z_{\Pi_0}^{\Pi'})\, dz_{\Pi_0}^{\Pi'} = \int\limits_{\Delta_{\Pi_0-\Pi}^{\Pi''} \backslash Z_{\Pi_0-\Pi}^{\Pi''}} \int\limits_{\Delta_{\Pi_0}^{\Pi} \backslash \Pi_0^{\Pi}} f(z_{\Pi_0}^{\Pi} z_{\Pi_0-\Pi}^{\Pi''})\, dz_{\Pi_0}^{\Pi}\, dz_{\Pi_0-\Pi}^{\Pi''}. \tag{10}$$

By virtue of this relation we have for every function $f \in H_\Pi$:

$$\int\limits_{\Delta_{\Pi_0}^{\Pi'} \backslash Z_{\Pi_0}^{\Pi'}} f(z_{\Pi}^{\Pi'} z)\, dz_{\Pi_0}^{\Pi} = \int\limits_{\Delta_{\Pi_0-\Pi}^{\Pi''} \backslash Z_{\Pi_0-\Pi}^{\Pi''}} f(z_{\Pi_0-\Pi}^{\Pi''} z)\, dz_{\Pi_0-\Pi}^{\Pi''}. \tag{11}$$

From (11) it follows that the conditions (7) and (9) are equivalent.

We shall prove the following theorem.

THEOREM. *The spaces H_Π are pairwise orthogonal and their direct sum is the whole space H, that is,*

$$H = \sum_{\Pi} H_\Pi$$

Here Π ranges over all subsets of Π_0, including Π_0 itself and the empty set.†

As a preliminary, we introduce the operators P_α that associate with every function $f \in H$ its integral over the Π-horospheres:

$$P_\Pi f(z) = \int\limits_{\Delta_{\Pi}^{\Pi} \backslash Z_{\Pi_0}^{\Pi}} f(z_{\Pi_0}^{\Pi} z)\, dz_{\Pi_0}^{\Pi}. \tag{12}$$

Obviously, P_Π is a projection operator, projecting the whole space H onto the subspace H'_Π. Thus, in terms of the operators P_Π the subspaces H'_Π and H_Π can be described in the following way:

1. H'_Π is the subspace of all functions $f \in H$ such that $P_\Pi f = f$.

2. H_Π is the subspace of all functions $f \in H$ such that $P_\Pi f = f$ and $P_{\Pi'} f = 0$ for $\Pi' \supset \Pi$.

Now we show that the *operators P_Π satisfy the following relation*:

$$P_{\Pi_1} P_{\Pi_2} = P_{\Pi_1 + \Pi_2} \tag{13}$$

for arbitrary subsets Π_1 and Π_2 of simple roots.

For the proof we use the following easily verifiable integral relation analogous to (10).

† Note that $H = H'_0$ and that $H'_{\Pi_0} = H_{\Pi_0}$ is the one-dimensional space of constants.

Let $f(z)$ be an arbitrary summable function on the group $Z^{\Pi}_{\Pi_0}$ that is constant on the cosets of $\Delta^{\Pi}_{\Pi_0} \setminus Z^{\Pi}_{\Pi_0}$. Then

$$\int_{F^{\Pi}_{\Pi_0}} f(z)\, dz = \int_{F^{\Pi_0-(\Pi+\Pi')}_{\Pi_0-(\Pi+\Pi')}} \int_{F^{\Pi,\Pi'}_{\Pi_0}} f(z'z'')\, dz''\, dz', \tag{14}$$

where

$$F^{\Pi}_{\Pi_0} = \Delta^{\Pi}_{\Pi_0} \setminus Z^{\Pi}_{\Pi_0},\ F^{\Pi,\Pi'}_{\Pi_0} = (\Delta^{\Pi}_{\Pi_0} \cap \Delta^{\Pi'}_{\Pi_0}) \setminus (Z^{\Pi}_{\Pi_0} \cap Z^{\Pi'}_{\Pi_0}).$$

The operator $P_{\Pi_1}P_{\Pi_2}$ is given by the following formula:

$$P_{\Pi_1}P_{\Pi_2}f(z) = \int_{F^{\Pi_1}_{\Pi_0}} \int_{F^{\Pi_2}_{\Pi_0}} f(z^{(2)}z^{(1)}z)\, dz^{(2)}\, dz^{(1)}. \tag{15}$$

By (14), the integral (15) can be represented in the form

$$P_{\Pi_1}P_{\Pi_2}f(z) = \int_{F^{\Pi_0-(\Pi_1+\Pi_2)}_{\Pi_0-(\Pi_1+\Pi_2)}} \int_{F^{\Pi_1,\Pi_2}_{\Pi_0}} \int_{F^{\Pi_2}_{\Pi_0}} f(z^{(2)}z'z''z)\, dz^{(2)}\, dz'\, dz''.$$

Since $Z^{\Pi_1}_{\Pi_0} \cap Z^{\Pi_2}_{\Pi_0} \subset Z^{\Pi_2}_{\Pi_0}$, we may change the variable on the right by setting $z^{(2)} = \hat{z}^{(2)}z'^{-1}$. As a result of this change we obtain

$$P_{\Pi_1}P_{\Pi_2}f(z) = \int_{F^{\Pi_0-(\Pi_1+\Pi_2)}_{\Pi_0-(\Pi_1+\Pi_2)}} \int_{F^{\Pi_2}_{\Pi_0}} f(z^{(2)}z''z)\, dz^{(2)}\, dz'',$$

that is, by (14),

$$P_{\Pi_1}P_{\Pi_2}f(z) = \int_{F^{\Pi_1+\Pi_2}_{\Pi_0}} f(z'z)\, dz' = P_{\Pi_1+\Pi_2}f(z).$$

So we have shown that $P_{\Pi_1}P_{\Pi_2} = P_{\Pi_1+\Pi_2}$.

On the basis of this result we can now prove the first assertion of the theorem, that the spaces H_{Π} are pairwise orthogonal.

Let $f_1 \in H_{\Pi_1}, f_2 \in H_{\Pi_2}$, $\Pi_1 \neq \Pi_2$. Then $f_1 = P_{\Pi_1}f_1, f_2 = P_{\Pi_2}f_2$. Consequently,

$$(f_1, f_2) = (P_{\Pi_1}f_1, P_{\Pi_2}f_2) = (f_1, P_{\Pi_1}P_{\Pi_2}f_2) = (f_1, P_{\Pi_1+\Pi_2}f_2).$$

But by the definition of the space H_{Π_2} we have $P_{\Pi_1+\Pi_2}f_2 = 0$, hence, $(f_1, f_2) = 0$, as required.

Now we prove the second assertion of the theorem, namely that

$$\sum_{\Pi} H_{\Pi} = H.$$

The proof is by induction on the number of simple roots.

From the definition of H'_{Π} and H_{Π} it follows that

$$H = H'_0 = H_0 + \left(\bigcup_{\alpha\in\Pi_0} H'_{\{\alpha\}}\right), \tag{16}$$

where 0 denotes the empty set and $\{\alpha\}$ the set consisting of the simple root α only. Thus, it is sufficient to show that

$$H'_{\{\alpha\}} \subset \sum H_{\Pi}.$$

Let us study the space $H'_{\{\alpha\}}$ in more detail. This space consists of the functions $f(z)$ that are constant on the $\{\alpha\}$-horospheres, that is, such that

$$f(z^{\{\alpha\}}_{\Pi_0} z) = f(z) \qquad \text{for every} \qquad z^{\{\alpha\}}_{\Pi_0} \in Z^{\{\alpha\}}_{\Pi_0}. \tag{17}$$

Since the group Z splits into the semidirect product

$$Z = Z^{\{\alpha\}}_{\Pi_0} Z^{\Pi_\alpha}_{\Pi_\alpha}, \qquad \text{where} \qquad \Pi_\alpha = \Pi_0 - \{\alpha\},$$

$H'_{\{\alpha\}}$ may be regarded as a space of functions on $\Delta^{\Pi_\alpha}_{\Pi_\alpha} \setminus Z^{\Pi_\alpha}_{\Pi_\alpha}$:

$$H'_{\{\alpha\}} = L_2(\Delta^{\Pi_\alpha}_{\Pi_\alpha} \setminus Z^{\Pi_\alpha}_{\Pi_\alpha}).$$

Note that $Z^{\Pi_\alpha}_{\Pi_\alpha}$ is a regular group whose set of simple roots Π_α is smaller than the set Π_0 of simple roots of the original group Z. Consequently, by the inductive hypothesis, the assertion of the theorem is true for the space $H'_{\{\alpha\}}$.

So we have

$$H'_{\{\alpha\}} = \sum_{\Pi \subset \Pi_\alpha} H_{\alpha,\Pi}, \tag{18}$$

where $H_{\alpha,\Pi}$ is the subspace of functions on $Z^{\Pi_\alpha}_{\Pi_\alpha}$ that satisfy the following conditions:

$$f(z^{\Pi}_{\Pi_\alpha} z) = f(z) \qquad \text{for every} \qquad z^{\Pi}_{\Pi_\alpha} \in Z^{\Pi}_{\Pi_\alpha}; \tag{19}$$

$$\int_{\Delta^{\Pi'}_{\Pi_\alpha} \setminus Z^{\Pi'}_{\Pi_\alpha}} f(z^{\Pi'}_{\Pi_\alpha} z)\, dz^{\Pi'}_{\Pi_\alpha} = 0 \qquad \text{for} \qquad \Pi_\alpha \supseteq \Pi' \supset \Pi. \tag{20}$$

The functions $f(z)$ may be assumed to be extended to the whole group Z according to (17).

We show that $H_{\alpha,\Pi} \subset H_{\Pi+\{\alpha\}}$. For let $f \in H_{\alpha,\Pi}$. Since $Z^{\Pi+\{\alpha\}}_{\Pi_0} = Z^{\{\alpha\}}_{\Pi_0} Z^{\Pi}_{\Pi_\alpha}$, it follows from (17) and (19) that $f(z^{\Pi+\{\alpha\}}_{\Pi_0}) = f(z)$ for every $z^{\Pi+\{\alpha\}}_{\Pi_0} \in Z^{\Pi+\{\alpha\}}_{\Pi_0}$, that is, the function f is constant on the $(\Pi + \{\alpha\})$-horospheres. Next, let $\Pi' \supset \Pi + \{\alpha\}$, that is,

$$\Pi' = \Pi'_1 + \{\alpha\},$$

where $\Pi_\alpha \supseteq \Pi'_1 \supset \Pi$. Since $Z^{\Pi'}_{\Pi_0} = Z^{\{\alpha\}}_{\Pi_0} Z^{\Pi'_1}_{\Pi_\alpha}$, we have by (17) and (20):

$$\int_{\Delta^{\Pi'}_{\Pi_0} \setminus Z^{\Pi'}_{\Pi_0}} f(z^{\Pi'}_{\Pi_0} z)\, dz^{\Pi'}_{\Pi_0}$$

$$= \int_{\Delta^{\{\alpha\}}_{\Pi_0} \setminus Z^{\{\alpha\}}_{\Pi_0}} \int_{\Delta^{\Pi'_1}_{\Pi_\alpha} \setminus Z^{\Pi'_1}_{\Pi_\alpha}} f(z^{\{\alpha\}}_{\Pi_0} z^{\Pi'_1}_{\Pi_\alpha} z)\, dz^{\Pi'_1}_{\Pi_\alpha}\, dz^{\{\alpha\}}_{\Pi_0}$$

$$= \int_{\Delta^{\Pi'_1}_{\Pi_\alpha} \setminus Z^{\Pi'_1}_{\Pi_\alpha}} f(z^{\Pi'_1}_{\alpha} z)\, dz^{\Pi'_1}_{\Pi_\alpha} = 0.$$

So we have shown that $H_{\alpha,\Pi} \subset H_{\Pi+\{\alpha\}}$. By (18) and (16) we find that $\sum_{\Pi} H_\Pi = H$. This completes the proof of the theorem.

GUIDE TO THE LITERATURE

Chapter 1, § 1. **§ 1** contains an account of essentially classical results. The theorem on the finiteness of the number of sides of a fundamental domain on the Lobachevskii plane and the existence of parabolic vertices is due to Siegel.

Chapter 1, § 2. Frobenius introduced the concept of induced representations for finite groups [17]. The importance of their role in the theory of infinite-dimensional representations of groups was discovered by Gel'fand and Naimark [29, 30]. The detailed theory of induced representations was developed by Mackey [49]. The theorem on the discreteness of the spectrum of an induced representation in the case of a compact space $\Gamma \setminus G$ is due to Gel'fand and Pyatetskii-Shapiro [31]. The criterion for complete reducibility of a representation given on p. 26 was found by Fell [16]. The trace formula (**4** and **5**) for the general case was obtained by Gel'fand and Pyatetskii-Shapiro [31]. The special case that refers to spaces on which the Laplace operators commute was obtained earlier, in another form, by Selberg [62].

Chapter 1, § 3. The irreducible unitary representations of the group of real matrices of order 2 were classified by Bargmann [2]. The spaces Ω_s (**§ 3.5**) were introduced by Gel'fand and Pyatetskii-Shapiro [31].

Chapter 1, § 4. The duality theorem for the representations of the discrete series of the group of matrices of order 2 was found by Gel'fand and Fomin [19]. The duality theorem for representations of the continuous series of the group of matrices of order 2 was obtained by Gel'fand and Pyatetskii-Shapiro [31]. Also in this paper is the general duality theorem of Pyatetskii-Shapiro. Property 1 of semisimple Lie groups (**§ 4.6**) was established by Harish-Chandra.

Chapter 1, § 5. The trace formula for the group of real matrices of order 2 was obtained, by other methods, by Selberg and is known in the literature as Selberg's trace formula [62]. The trace formula for the group of complex matrices of order 2 (**§ 5**, App. **2**) was first published in the paper by Gel'fand and Pyatetskii-Shapiro [31]. The theorem on continuous deformations (**§ 5**, App. **1**) is also from Gel'fand and Pyatetskii-Shapiro.

Chapter 1, § 6. The results (for any semisimple group G and a so-called regular discrete subgroup) are from Gel'fand and

Pyatetskii-Shapiro [34]. The concept of a horospherical map, its applications to representation theory, and the clarification of its connection with integral geometry are due to Gel'fand and Graev [22]. An account of some related problems is given in [28]. The most recent results in this field were obtained by Helgason [43]. The application of the horospherical map to the study of representations in the spaces $L_2(\Gamma \backslash G)$ is due to Gel'fand and Pyatetskii-Shapiro [32, 33, 34].

Chapter 1, Appendix. Quaternion groups were studied systematically by Eichler.

Chapter 2, § 2. The spaces of test and generalized functions on a locally compact group were introduced by Bruhat [12]. The concepts of the Gamma-function, Beta-function, Bessel function and hypergeometric function for every locally compact field are due to Gel'fand and Graev [26].

Chapter 2, § 3. The first papers on the theory of unitary representations of matrix groups with p-adic elements are by Mautner [50] and Bruhat [11]. The results of **1** through **9** are due to Gel'fand and Graev [26]. The formula for spherical functions in **10** was found by Mautner [50]. Spherical functions in the more general case were studied by Satake [60]. The results of **11** are due to the authors.

Chapter 2, § 4. The results of **1** through **7** are due to Gel'fand and Graev [26]; those of **8** to Kirillov.

Chapter 2, § 5. The trace formulae for a disconnected field were obtained by Gel'fand and Graev [26], for the field of complex numbers by Gel'fand and Naimark [30], and for the field of real numbers by Harish-Chandra [39].

Chapter 2, § 6. The Plancherel formula for a disconnected field was obtained by Gel'fand and Graev [26], for the field of complex numbers by Gel'fand and Naimark [30], and for the field of real numbers by Bargmann [2]; see also Harish-Chandra [39].

Chapter 2, Appendix. The results are due to Kirillov, who also wrote the text.

Chapter 3, § 1. The concepts of an adele and idele are due to Chevalley; see also Tate (thesis), Weil [72] and Lang [46].

The Appendix to 1 is due to Pyatetskii-Shapiro. Similar results were obtained by Satake.

Chapter 3, § 2. The main results are due to Tate (thesis). The concept of a Tate ring (**§ 2**, App.) is due to the authors.

Chapter 3, § 3. The concept of the adele group of an arbitrary group G is due to Ono [54, 55], Tamagawa and, for the case of the orthogonal group, to Kneser [45]. The concept of the tensor product of representations for the direct product of groups with distinguished subgroups, which is introduced here, is due to the authors of the book [27]. The first theorem on tensor products in **2** and **3** is by the authors. The second theorem in **5** was obtained by the authors together with Kirillov. The criteria for the existence of a unique invariant vector were found essentially by Gel'fand [18].

Chapter 3, § 4. The method expounded in the book is that of the authors. The Russian edition of the book contained errors that were pointed out to the authors by Godement.† In the present English edition the authors have largely rewritten the text and eliminated the mistakes. The results of **§ 4**, Appendix II are due to Pyatetskii-Shapiro. Similar results were obtained by Satake.

Chapter 3, §§ 5 and 6. The main results are due to the authors. Considerably more complete results, even for an arbitrary reductive group, were obtained by Langlands (preprint). A brief account of his results is given in his paper "Eisenstein series" [47], and in Godement's survey [37].

Chapter 3, § 7. The main theorem is due to the authors; the main lemma to Gel'fand and Pyatetskii-Shapiro [34]. **§ 7** Appendix is due to the authors.

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† Godement also communicated (in a letter to the authors) his own elegant way of avoiding these errors.

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Index of Names

Subject Index